Introduction

The title of the thesis is "Structure of finite perimeter sets in Carnot groups" and its main purpose is to present some recent results of Geometric Measure Theory in Carnot groups. Here we give a brief introduction of this topic and an overview of the thesis at the end of this chapter.

Recently a great amount of effort were made to generalize classical methods of Analysis, such as Sobolev spaces [30], topics of Geometric Measure Theory such as rectifiability and currents ([1], [2], [3], [33],[27], [28]), to general metric spaces. First we want to outline the historical grow of this research focusing our attention to the so-called Carnot-Carathéodory spaces, that in the sequel we will abbreviate with CC-spaces, and in particular to Carnot groups.

Carnot-Carathéodory spaces appear, in the literature, under a variety of names such as Sub-Riemannian spaces, nonholonomic Riemannian spaces, singular Riemannian spaces ([27], [32], [53]). Historically this subject started with the 1909 work of C. Carathéodory [8] on the second principle of Theormodynamics: he represented a thermodynamic process by a curve in $\mathbb{R}^n$ and the heat exchanged during it is represented by the integral of this curve along a suitable 1-form θ. Using this representation, J. Carnot proved the existence of two states that cannot be connected by an adiabatic process, i.e. θ vanish at every point. Such curves are nowadays called "horizontal", i.e. curves whose velocity belongs to a suitable subspace of the whole tangent bundle. Later Carathéodory proved that if there exist two point not connectible by an horizontal curve, then θ is integrable, this means that there exist two functions T and S such that $\theta = T\,dS$. If we interpret respectively T and S as the temperature and the entropy, the previous formula leads to the mathematical formulation of the second principle of Thermodynamic. The Carathéodory result can be seen from an other point of view: if θ is a non-integrable 1 form, then we can connect any two points by an horizontal curve.

In a more general setting the problem of connecting points by means of horizontal curves was solved independently by P. K. Rashevsky [46] and W.L. Chow [12]; they prove a sufficent condition for the connectivity the so called “Chow condition”. Representing the horizontal distribution by a system of differential fields, if at any point the Lie algebra generated by this vector fields has the same dimension of the tangent space, we say that the “Chow condition” is satisfied. This condition, known also as “Hörmander condition” or “bracket generating condition”, has played a role in different fields of mathematics such as Optimal Control Theory and Subelliptic PDE’s ([20], [21], [22]). Reconsidering the previous example, the integrability of θ implies that kerθ does not generate the whole Lie algebra, thus it doesn’t satisfy the “Chow condition”.

Consider a manifold M and a sub-bundle, called “horizontal”, of the tangent bundle. We say that a curve γ is horizontal if it is tangent to the horizontal boundle. If the “Chow condition” is satisfied we can define a distance associated to the horizontal boundle. Define the distance d_c between $p, q \in M$ as the infimum of lengths of all horizontal curves connecting the two pont, the bracket generating condition ensure that d_c is always finite. The distance d_c is the so-called Carnot-Carathéodory distance ([44], [27]) and M togheter with the horizontal boundle is called a CC space. Given a CC-space (M, d_c) and $p \in M$ we can define the tangent space to M at p in the sense of Gromov [7] as $(T_pM, 0) = \lim_{\lambda\to\infty}(\lambda M, p)$, it is not difficult to prove that when M is a Riemannian manifold the tangent space in the metric sense coincide with the classical one. If, instead, we consider a general CC-space the limit has a natural group structure, but can be really different from the tangent space of a smooth manifold. The first work in this direction is due to Mitchell, recently refined by Bellaïche [7] and Margulis and Mostow [38], where it is proved that the tangent space to a CC-space is a simply connected nilpotent Lie group, where its Lie algebra is graded and generated by its component of degree 1. Such space is called Carnot group by Pansu in [44], thus we can consider Carnot groups as a local model of CC-spaces.

Carnot group, the main topic of this thesis, are finite dimensional connected and simply connected Lie groups $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is stratified, this means that $\mathfrak{g}$ can be written as

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_s$$

where V_i are subspaces of $\mathfrak{g}$ with the property $V_{i+1} = [V_i, V_1]$ and $V_j = \{0\}$ if $j > s$, s is called the step of the group. Such groups are examples of CC-spaces, on them we can define a CC-distance d_c associated to the fields $(X_1, \ldots, X_m)$, here $(X_1, \ldots, X_m)$ is a basis of the first layer V_1 of the Lie algebra $\mathfrak{g}$. Given a Carnot group $\mathbb{G}$ one can define naturally a one parameter family of dilations δ_λ, it results that d_c is homogeneus with respect to the dilations δ_λ and invariant under translations, i.e.

$$d_c(\delta_\lambda p, \delta_\lambda q) = \lambda d_c(p,q) \qquad d_c(zp, zq) = d_c(p,q) \quad \forall \lambda > 0,\ \forall z \in \mathbb{G}.$$

Given the CC-distance d_c, we can build the k-Hausdorff measure $\mathcal{H}^k$ associated, it follows, using the ball-box Theorem of Nagel,Stein and Wainger [50], that the Hausdorff measure of $\mathbb{G}$ is exactly $Q = \sum_{i=1}^m i\, dim V_i$ (the so-called homogeneus dimension). Notice that if $\mathbb{G}$ is a group of step $s > 1$ the Hausdorff dimension is strictly greather that the topological one, this fact shows how Carnot groups are far from being Euclidean spaces.
Moreover on every Carnot group is always defined a left invariant Haar measure $vol_{\mathbb{G}}$, by the property of the CC-distance d_c and the uniqueness of the Haar measure it follows that $\mathcal{H}^Q$ is a scalar multiple of the Haar measure. It is not difficult to prove that $\mathbb{G}$ is an Ahlfors Q-regular metric space, i.e. there exists a constant $a > 0$ such that

$$\frac{1}{a}\rho^Q \leq vol_{\mathbb{G}}(B_\rho(x)) \leq a\rho^Q \qquad \forall x \in \mathbb{G},\ \forall \rho > 0.$$

Now we open a parenthesis to present some results of Geometric Measure Theory in metric spaces, these results will play an important role in the sequel. We need to mention the works of Miranda [41], Ambrosio [2], Kirchheim [33], Cheeger [10], Cheeger and Kleiner [11], Ambrosio, Miranda and Pallara [2], but this list is far from being complete.
In the paper [41], Miranda extended the notion of function of bounded variation and set of finite perimeter to “good” metric measure spaces (X, d, μ). Here “good” means that μ satisfies a doubling property, i.e. $\exists C > 0$ such that

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)) \qquad \forall x \in X,\ \forall r > 0,$$

and X supports a 1-Poincaré inequality, i.e. given $u \in Lip_{loc}(X)$ there holds

$$\int_B \|u(x) - u_B\| d\mu(x) \leq Cr(B) \int_{\lambda B} g(x) d\mu(x) \qquad for\ any\ ball\ B \subseteq X$$

where $\lambda \geq 1$ is a suitable constant and g is an Upper gradient of u (see [30]).
Carnot groups, as Nagel, Stein and Wainger proved in [43] for general CC-spaces, are doubling metric spaces. Moreover they support a 1-Poincaré inequality (see [30]), thus one can define on them BV functions and finite perimeter sets as in [41]. Indeed Carnot groups have a richer structure than a general "good" metric space (e.g. the graded structure and the homogeneus dilations), thus we can give a less abstract definition of set of locally finte perimeter. Consider a Carnot group $\mathbb{G}$, we first define the X derivative of a $L^1_{loc}(\mathbb{G})$ function. Given a vector field $Y \in \Gamma(\mathbb{G})$ we define the divergence $divY$ as

$$\int_{\mathbb{G}} Yu d\, vol_{\mathbb{G}} = -\int_{\mathbb{G}} u divY d\, vol_{\mathbb{G}} \qquad \forall u \in C^\infty_c(\mathbb{G}),$$

if Y is divergence-free and $u \in L^1_{loc}(\mathbb{G})$, we call X-derivative the distribution

$$\langle Xu, v\rangle = -\int_{\mathbb{G}} uXv\, d\, vol_{\mathbb{G}} \qquad \forall v \in C^\infty_c(\mathbb{G}).$$

Recall that the vector fields $(X_1, \dots, X_m)$ span the first layer of $\mathfrak{g}$, consider a set $E \subset \mathbb{G}$ and suppose that $D\chi_E = (X_1\chi_E, \dots, X_m\chi_E)$ is a vector-valued Radon measure, we say that E is a set of locally finite perimeter ad set $|D\chi_E|$ its total variation. As a consequence of the general theory in doubling metric spaces, it follows that $|D\chi_E|$ satisfies almost all the classical property of the perimeter measure in Euclidean spaces. Recall that $\mathbb{G}$ is an Ahlfors space then, by Theorem (4.2) in [5], follows that $|D\chi_E|$ is concentrated on the measure theoretic boundary $\partial^* E$, defined as in the Euclidean case. Moreover, see [1], the following representation formula with respect to the Hausdorff measure $\mathcal{H}^{Q-1}$ holds

$$|D\chi_E|(A) = \int_{\partial^* E \cap A} \theta_E(g)\, d\mathcal{H}^{Q-1} \qquad \forall A \in \mathcal{B}(\mathbb{G}). \tag{1}$$

where θ is a suitable borel function.
Following the idea of De Giorgi [16], we can analyse the structure of the tangent set, see Definition (3.8). One of the first works in this direction is the paper of B. Franchi, R. Serapioni and F. Serra Cassano [24], where they give a complete descriprion of the tangent set of locally finite perimeter sets in step 2 groups. For general Carnot groups, there is not yet a satisfactory answer to this problem, neverthless recently L. Ambrosio, B. Kleiner and E. Le Donne [4] proved the existence, $\mathcal{H}^{Q-1}$-a.e. on $\partial^* E$, of a vertical halfspace (see 3.16) in the tangent space, here $E \subset \mathbb{G}$ is a set of locally finite perimeter. Generalizing the notion of tangent space to couple of sets and using the same techniques of [4] we are

able to prove the locality property of Carnot groups and we say, as in [5], that $\mathbb{G}$ is a $\mathcal{U}$-space. This means that given two sets $E, F \subset \mathbb{G}$ of locally finite perimeter then

$$\theta_E(x) = \theta_F(x) \qquad for\ \mathcal{H}^{Q-1} - a.e. \quad x \in \partial^* E \cap \partial^* F,$$

where θ_E is the Borel function in the representation formula (1). As a consequence of the locality property, we can give a simplified formula for the total variation of a function in $BV(\mathbb{G})$ or in $SBV(\mathbb{G})$, as proved in [5]. Moreover, one can show the lower semicontinuity of a Munford-Shah type functional, for every Carnot group. This extends previous results in the Heisenberg group by Song and Yang [48], Citti, Manfredini and Sarti [13].

Another interesting problem which has been deeply studied is the definition of rectifiability in CC-spaces and in particular in Carnot groups. The classical definition of rectifiability in metric spaces given by Federer in [19], i.e rectifiable sets are images of subsets of Euclidean spaces via Lipschitz maps, does not suit the geometry of Carnot groups. Indeed, with this definition of rectifiability, the Heisenberg group $\mathbb{H}^1$ is purely k-unrectifiable for $k = 2, 3, 4$, see [3] for the proof and [37] for a generalization. An intrinsic definition of rectifiability is given in the remarkable paper of B. Franchi, R. Serapioni and F. Serra Cassano [23]. They define $\mathbb{H}$-surfaces as level set of functions $f : \mathbb{H}^n \to \mathbb{R}$ whose horizontal derivative is continuous and nonvanishing, and $\mathbb{H}$-rectifable sets as sets contained, up to a negligible sets, in a countable union of $\mathbb{H}$-regular surfaces. Moreover in [23] it is proved that the perimeter measure of a locally finite perimeter set is concentrated on a $\mathbb{H}$ rectifiable set. As a consequence of the paper [24] it follows that the rectifiability results can be generalized to all step 2 Carnot groups, the general case of groups of step > 2 being still open.

Overview of the thesis

In *Chapter* 1 we present some recent results of Geometric Measure Theory in doubling metric measure spaces and in Ahlfors k-reguar spaces, we define the class of functions of Bounded Variation and the sets of finite perimeter, following the the paper of Miranda [41]. In the second section we state some results of the theory of finite perimeter sets in Ahlfors spaces, contained in [1]. Those theorems will play and important role when dealing with Carnot groups, that are a particular example of Ahlfors spaces. In the last section we focus our attention to the Special functions of Bounded Variation, we prove some theorems on the structure of the discontinuous set of a BV function and we define the $\mathcal{U}$-spaces.

In *Chapter* 2 we state the main features abount CC-spaces. In section (2.1) we recall the definition of Lie group and Lie algebra and we prove the existence of a diffeomorphism between a nilpotent Lie group and its Lie algebra. In section (2.2) we give the definition of CC-distance, we state the Theorem of Chow and, following [7], we analize the structure of the tangent set to a CC-space. In section (2.3) we prove the existence (see [30]) of the minimal upper gradient of a continuous function in CC-spaces, continuous with respect to the CC distance.

Chapter 3 is the core of the thesis, here we begin giving the definition of a Carnot group and proving some basic properties of the CC-distance associated to a basis of the first layer of the Lie algebra. Section (3.2) is devoted to the exposition of the results contained in [4], where it is proved the existence of a vertical halfspaces in the tangent set to a Carnot group. In section (3.3) we generalize some method of [4] to prove the locality property (see 1.11) for every Carnot group. Finally, in section (3.4) we present the rectifiability problem in Carnot groups and we state the main results of [24].

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Contents

Chapter 1

BV functions in metric spaces

In this chapter we present some recent results of Geometric Measure Theory on Metric spaces that will be used in the sequel when dealing with CC-spaces and Carnot groups. The first section is devoted to the study of Banach space valued functions with bounded variations. The metric space we consider is doubling and support a Poincaré inequality for suitable couples of functions. One can prove (see [10]) that a space with these properties is almost geodesic, i.e. there exists a geodesic metric that is Lipschitz equivalent to the original one. Examples of these spaces are CC-spaces (see Chapter 2) and Carnot groups, the main topic of the thesis.
In $\mathbb{R}^n$, given an open set $\Omega \subset \mathbb{R}^n$, $u \in L^1(\Omega)$ is said to have bounded variation, or $u \in BV(\Omega)$, if

$$\|Du\|(\Omega) := \sup \left\{ \int_\Omega u \, div\phi dx \ \phi \in C_0^1(\Omega, \mathbb{R}^n), \quad \|\phi\|_\infty \leq 1 \right\} < \infty$$

An equavalent definition, as one can see for example in [18], is the following: a function u is in $BV(\Omega)$ if and only if there exists a constant $M > 0$ and $(u_h)_h \in C_0^1(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$ and

$$\limsup_{h\to\infty} \int_\omega \|\nabla u_h\| dx \leq M.$$

This definition make sense also in metric spaces if we replace smooth functions by Lipschitz one. Following [41], we will prove some property of BV functions and we will introduce the theory of finite perimeter sets in doubling metric spaces.
In the second section we consider Ahlfors k-regular spaces, i.e. metrics measure spaces

(X, d, μ) such that there exists $C > 0$

$$1/C\rho^k \leq \mu(B_\rho(x)) \leq C\rho^k \qquad \forall x \in X,\ \rho \in (0, diamX).$$

These spaces are examples of doubling metric measure spaces, moreover if we suppose that a Poincaré inequality holds, we can define by a relaxation procedure, as we have seen in the previous section, the class of BV functions. Here we present the main results of the paper of L. Ambrosio [1]. Following the idea of De Giorgi ([16]) on the rectifiability theorem for Euclidean sets he proves a representation formula of the perimeter measure with respect to the Hausdorff $(k-1)$ dimensional measure and a density estimates on the volume and perimeter. As a consequence of these results it will follows the asymptotically doubling property of the perimeter measure, this implies that the spherical differentiation theory can be done using the perimeter measure. In Chapter 3 this result will be foundamental for the study of the Tangent set to a Carnot group.
In the last section following [2], it is given an extension of special function of bounded variation to doubling metric spaces. Here it is proved an upper and lower bound on the function θ_E that represent the perimeter measure, of a set E, with respect to the spherical measure $\mathcal{S}^h$, a chain rule for BV functions and the closure and compactness theorems. Finally we define local spaces or $\mathcal{U}$-spaces and we state, in this setting, the existence of a minimizer for a Munford-Shah type functional.

1.1 Doubling metric measure spaces

Throughout this section, we will consider a special class of metric spaces, the doubling metric measure spaces. In this setting we can prove the classical Vitali covering theorem, and a Lebesgue type differentiation therem that can be used when a Besicovich differentiation theorem doesn't hold. We will follow the paper of Miranda [41].

1.1.1 Doubling spaces and Poincaré inequality

Definition 1.1 (Doubling space)**.** A doubling space is a complete metric measure space (X, d, μ) such that

$$\mu(2B) \leq c\mu(B) \quad for\ every\ ball\ B\ \in \mathcal{B} \tag{1.1}$$

for some costant $c \geq 0$. The best costant c_D in the previous inequality is called the doubling constant for μ.

Remark 1. There exist a lower bound for the density of the space X, i.e., if we set $s = log_2 C_D$

$$\frac{\mu(B_\rho(x))}{\mu(B_R(x))} \geq \frac{1}{C_D^2}\left(\frac{\rho}{F}\right)^s, \qquad \forall\, 0 < \rho \leq R < \infty, \quad x, y \in X. \tag{1.2}$$

Example 1. If we take $X = \mathbb{R}^N$, $d(x,y) = |x - y|$ the Euclidean metric and $\mu = \mathcal{L}^N$, the Lebesgue measure, then $(\mathcal{R}^N, |\cdot|, \mathcal{L}^N)$ is a doubling metric measure space with $C_D = 2^N$.

Example 2. Consider $X = (M, g)$ a complete Riemannian manifold of dimension N and μ the canonical volume measure associated to the metric g. Then if the Ricci curvature is nonnegative (see [9]), (X, g, μ) is a doubling measure space with $C_D = 2^N$.

Example 3. ([1]) Consider the space $X = [-1,0] \times [-1,1] \cup [0,1] \times 0$, d the Euclidean metric and $\mu = \mathcal{L}^2 \llcorner X + \mathcal{H}^1 \llcorner [0,1] \times 0$, then μ is doubling with $C_D = 4$. Note that the dimension of the space (X, d, μ) is not costant.

As mentioned before a Vitali covering theorem holds. For the proof see [29].

Theorem 1.1.1 (Vitali covering theorem). *Let A be a subset in a doubling metric measure space (X, μ), and let $\mathcal{F}$ be a collection of closed balls centered in A such that*

$$\inf\{r > 0 : \ B(a,r) \in \mathcal{F}\} = 0 \quad \forall\ a\ \in A$$

Then, there is a countable disjointed subfamily $\mathcal{G}$ of $\mathcal{F}$ such that the balls in $\mathcal{G}$ cover μ a.e. of A, i.e.

$$\mu\left(A \setminus \bigcup_{\mathcal{G}} B\right) = 0.$$

We give now the proof, for completeness, of the Lebesgue differentiation theorem. It will be useful when dealing with perimeter measures. More generally this theorem allows to compute the density of the absolutely continuous part of a measure ν with respect to a doubling (or asymptotically doubling) measure μ, i.e. writing $\nu = f\mu + \nu^s$, with ν^s singular with respect to μ, we can compute

$$f(x) = \frac{d\nu}{d\mu}(x) = \lim_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \tag{1.3}$$

for μ-almost every $x \in X$.

Proposition 1.1.2. *If f is a nonnegative, locally integrable function on a doubling metric measure space (X, μ), then*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu = f(x) \tag{1.4}$$

for μ-almost every $x \in X$.

Proof. Let E denotes the set of points of X where Eq. (1.4) does not hold. We can find open sets $X_n \subset X$ such that $X \subseteq \bigcup X_n$ and $f \in L^1(X_n)$, so we can suppose that $f \in L^1(X)$. Thus, it suffices to show that E has measure zero in a fixed ball B.
To this end, we first claim that if $t > 0$ and if

$$\liminf_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu \leq t,$$

for each x in a subset A of B, then

$$\int_A f d\mu \leq t\mu(A). \tag{1.5}$$

To prove this claim, fix $\epsilon > 0$ and choose an open superset U of A such that $\mu(U) \leq \mu(A) + \epsilon$. Then, each point in A has arbitrarily small closed ball neighborhoods contained in U where the mean value of f is less than $t + \epsilon$. The Vitali covering theorem implies that we can pick a countable disjointed collection of such balls covering almost all of A, from which

$$\int_A f d\mu \leq (t + \epsilon)\mu(U) \leq (t + \epsilon)(\mu(A) + \epsilon),$$

and the claim follows upon letting $\epsilon \to 0$. A similar argument shows that if $t > 0$ and if

$$\limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu \geq t \tag{1.6}$$

for all $x \in A \subset B$, then

$$\int_A f d\mu \geq t\mu(A). \tag{1.7}$$

It follows, in particular, that

$$\limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu < +\infty$$

for almost every $x \in B$. On the other hand, if $A_{s,t}$ is the set of points in $x \in B$ for which

$$\liminf_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu \le s < t \le \limsup_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu,$$

using the Eq. (1.5) and Eq. (1.7) we get

$$t\mu(A_{s,t}) \le \int_{A_{s,t}} f d\mu \le s\mu(A_{s,t})$$

that imply $\mu(A_{s,t}) = 0$. Thus, the limit on the left in Eq. (1.4) exists and it is finte μ-almost everywhere in B. Denote this limit by $g(x)$ whenever it exist. It remains to show that $g(x) = f(x)$ μ-a.e. in B.

Fix a set $F \subset B$ and $\epsilon > 0$; for each integer n, denote

$$A_n = \{x \in F : \ (1+\epsilon)^n \le g(x) \le (1+\epsilon)^{n+1}\}.$$

Then, by Eq.(1.7)

$$\begin{aligned} \int_F g d\mu &= \sum_n \int_{A_n} g d\mu \le \sum_n (1+\epsilon)^{n+1} \mu(A_n) \\ &\le (1+\epsilon) \sum_n \int_{A_n} f d\mu = (1+\epsilon) \int_F f d\mu, \end{aligned}$$

and similarly we get

$$\begin{aligned} \int_F g d\mu &= \sum_n \int_{A_n} g d\mu \ge \sum_n (1+\epsilon)^{n} \mu(A_n) \\ &\ge (1+\epsilon)^{-1} \sum_n \int_{A_n} f d\mu = (1+\epsilon)^{-1} \int_F f d\mu. \end{aligned}$$

By letting $\epsilon \to 0$, we infer that

$$\int_F g d\mu = \int_F f d\mu$$

and that $g = f$ a.e. in B. The theorem follows. □

In section (1.2) we will prove that the perimeter measure $P(E, \cdot)$ satisfies a slightly different condition than the doubling property, i.e. the asymptotically doubling property.

Definition 1.2. Let μ a Borel measure on a metric space X, we say that μ is asymptotically doubling on X if for μ-a.e. $x \in X$ we have

$$\limsup_{\rho\to 0^+} \frac{\mu(B_{\lambda\rho}(x))}{\mu(B_\rho(x))} < \infty$$

for some $\lambda > 1$.

Following Theorem (2.8.17) of [19] we have

Theorem 1.1.3. *Let μ be an asymptotically doubling measure on X, which is finite on bounded sets. Then a Vitali covering theorem holds for μ.*

Remark 2. A differentiation theorem similar to Proposition 1.1.2 holds also with the doubling property replaced by the asympototically doubling property, for the proof we refer to [19].

Given a curve $\gamma : [0,1] \longrightarrow X$, we denote the length of γ by :

$$L(\gamma) = sup \ \Sigma \ d(\gamma(t_i), \gamma(t_{i-1})) \tag{1.8}$$

where the supremum is taken over all possible finite partitions $[t_{i-1}, t_i]$ of $[0,1]$. A curve γ joining x and y is called rectifiable if it has finite length, we denote any such curve by $\gamma : x \longrightarrow y$.

Definition 1.3 (Upper gradient)**.** Give a Lipschitz function $u : X \longrightarrow V$, an upper gradient for u is a Borel function $g : X \longrightarrow [0, +\infty]$ such that for every $x, \ y \ \in \ X$ and for every $\gamma : x \longrightarrow y$ there holds

$$|u(y) - u(x)| \leq \int_0^1 g(\gamma(s))|\gamma'|(s)ds,$$

where

$$|\gamma'|(s) = \liminf_{h \to 0} \frac{|\gamma(s+h) - \gamma(s)|}{|h|}.$$

If γ is a Lipschitz curve the limsup is at almost every point a limit, this limit is called the metric derivative of γ. We denote by $UG(u)$ the collection of all upper gradients of u.

Definition 1.4. A metric measure space (X, d, μ) is a Poincaré space if μ is a doubling measure and X support a weak Poincaré inequality, i.e., for every pair (u, g) with $u \in Lip_{loc}(X; V)$ and $g \in \ UG(u)$, there holds

$$\int_B |u(x) - u_B| d\mu(x) \leq Cr(b) \int_{\lambda B} g(x) d\mu(x) \quad for \ every \ ball \ B \subseteq X$$

for constants $\lambda \geq 1, \ C \geq 0$.

Example 4. Let consider the Euclidean space $\mathbb{R}^n$ endowed with the Lebesque measure $\mu = \mathcal{L}^n$. As stated before μ is a doubling measure, moreover a weak Poincaré inequality holds on $\mathbb{R}^n$, see [18].

Example 5. Consider $\mathbb{R}^n$, and fix $k < n$ vector fields $X = (X_1, \ldots, X_k)$, suppose that X satisfies the Hörmander's (or Chow's) condition, i.e. there exist p such that the family of commutators of the X_i of length p span $\mathbb{R}^n$ at every point. We say that a Lipschitz map $\gamma : [0,T] \to \mathbb{R}^n$ is *horizontal*, if there exist measurable functions $a_1, \ldots, a_k : [0,T] \to \mathbb{R}$ with $a_1^2 + \cdots + a_k^2 \leq 1$ and

$$\gamma'(t) = \sum_{i=1}^{k} a_i(t) X_i(\gamma(t)) \quad \mathcal{L}^1 - a.e.\ t \in [0,T]. \tag{1.9}$$

It si possible to define a distance on $(\mathbb{R}^n, X)$, called Carnot-Caratheodory metric, by setting

$$d(x,y) = \inf\left\{T : \exists\gamma : [0,T] \to \mathbb{R}^n \ as\ in\ Eq.(1.9),\ \gamma(0) = x,\ \gamma(T) = y\right\};$$

if there is no such curve, we set $d(x,y) = +\infty$.

If X satisfies the Chow's condition, every two points can be joined with an horizontal curve of finite length (see [7]) . Moreover, if u is a Lipschitz function, with respect to the Carnot-Caratheodory distance d, then

$$|Xu| = \sqrt{|X_1 u|^2 + \cdots |X_k u|^2}$$

is the minimal upper gradient for u (see [30]). An example of CC space is the Heisenberg group $\mathbb{H}^1$; it is $\mathbb{R}^3$ with the vector fields $X_1 = \partial_x + 2y\partial_z$, $X_2 = \partial_y - 2x\partial_z$ and $X_3 = \partial_z$, and Chow's condition for $\mathbb{H}^1$ holds. Moreover, $\mathbb{H}^1$ has a group structure, and it is the first example of a Carnot group. We will discuss in detail this kind of spaces in the next chapter.

1.1.2 Functions of bounded variation

Now we define the functions of bounded variation. In the Euclidean case ($\mathbb{R}^n$) we can give the following three equivalent definition of BV but only the last one will make sense in the generale case of metric spaces. Let $u \in L^1_{loc}(\Omega)$, $\Omega \subset\subset \mathbb{R}^n$ then the following conditions are equivalent :

1. $u \in BV(\Omega)$, i.e.
$$|Du|(\Omega) = \sup\left\{\int_\Omega u div\varphi dx : \ \varphi \in C_0^1(\Omega, \mathbb{R}^n), \ \|\varphi\| \leq 1\right\} < +\infty;$$
2. there exist a vector measure $\sigma = (\sigma_1, \ldots, \sigma_n)$ with finite total variation in $\mathbb{R}$ such that
$$\int_\Omega u\partial_i\varphi dx = -\int_\Omega \varphi d\sigma_i \quad \forall \varphi \in C_0^1(\Omega);$$
3. there esist a costant $M > 0$ and a sequence $(u_h)_h \subset C_0^1(\Omega)$ such that $u_h \to u$ in L^1 and
$$\limsup_{h\to\infty} \int_\Omega |\nabla u_h| dx \leq M.$$

Let $\Omega \subseteq X$ be an open set; since $Lip_{loc}(X; V)$ is dense in $L^1_{loc}(\Omega)$ we can define the total variation of a function in this metric setting, relaxing respect to the topology of $L^1_{loc}(\Omega)$ the functional
$$u \to \int_\Omega |\nabla u| d\mu,$$
here ∇u is an element of $UG(u)$.

Definition 1.5. For every $u \in L^1_{loc}(\Omega, \mu; V)$, the total variation of u on every open set $A \subseteq \Omega$ is
$$|Du|(A) = \inf\{\liminf_{h\to\infty} \int_A g_h d\mu : \ (u_h)_h \subseteq Lip_{loc}(A; V), \ u_h \xrightarrow{L^1_{loc}} u\},$$
where $g_h \in UG(u_h)$.

Definition 1.6. A function $u \in L^1_{loc}(\Omega, \mu; V)$ is said to have (locally) bonded total variation in Ω if $|Du|(\Omega) < \infty$ ($|Du|(A) < \infty \ \forall$ open set $A \Subset \Omega$). The vector space of function with (locally) bounded total variation will be denoted by $BV(\Omega, \mu; V)$ ($BV_{loc}(\Omega, \mu; V)$).

Remark 3. We have the following property of the set function $|Du|(\cdot)$; for every $u, v \in L^1_{loc}(\Omega, \mu; V)$, for every $\alpha \in \mathbb{R}$ and for every A, B open subsets of X.

1. $|D(\alpha u)|(A) = |\alpha| \cdot |Du|(A)$;
2. $|D(u + v)|(A) \leq |Du|(A) + |Dv|(A)$;

3. $|Du|(A \cup B) \geq |Du|(A) + |Du|(B)$ if $A \cap B = \emptyset$;

4. $|Du|(A \cup B) = |Du|(A) + |Du|(B)$ $dist(A, B) > 0$.

The first three points are direct consequences of the definitions. We prove only the last property : consider two sequences $(u_h)_h \in Lip_{loc}(A; V)$ and $(v_h)_h \in Lip_{loc}(B; V)$ converging to u in $L^1_{loc}(A, \mu; V)$ and $L^1_{loc}(A, \mu; V)$ respectively , s.t.

$$\lim_{h\to\infty} \int_A |\nabla u_h| d\mu = |Du|(A), \ \lim_{h\to\infty} \int_B |\nabla v_h| d\mu = |Dv|(B).$$

If we define :

$$w_h = \begin{cases} u_h \ on \ A, \\ v_h \ on \ B, \end{cases}$$

we have a new sequence converging to u let $p_h \in UG(w_h)$, using the fact that A and B are distant sets, we have

$$\begin{aligned} |Du|(A \cup B) &\leq \liminf_{h\to\infty} \int_{A\cup B} p_h d\mu = \\ \lim_{h\to\infty} \int_A |\nabla u_h| d\mu + \int_B |\nabla v_h| d\mu &= |Du|(A) + |Du|(B). \end{aligned} \tag{1.10}$$

The other inequality follow from (3), so we get equality in (4).

The next theorem tell us that $|Du|$ defines a measure, but it is not possible in this general context to use the Euclidean techniques such as Riez representation therem to prove it. The approach used in the proof in [41] is typical in the study of relaxations problems.

Theorem 1.1.4. *For any $u \in BV_{loc}(\Omega, \mu; V)$, the set function $|Du|$ is the restriction to the open subset of X of a positive locally finite measure in X.*

The following proposition can be proved by a diagonal method.

Proposition 1.1.5. *Let $\Omega \subseteq X$ be an open set and let $(u_h)_h$ be a sequnce in $BV_{loc}(\Omega, \mu; V)$ such that $u_h \to u$ in $L^1_{loc}(\Omega, \mu; V)$. Then*

$$|Du|(A) \leq \liminf_h |Du_h|(A) \quad for \ any \ open \ set \ A \subseteq \Omega.$$

In particular, if $sup_h |Du_h|(A) < +\infty$ for every open set $A \Subset \Omega$, the limit function u is in $BV_{loc}(\Omega, \mu; V)$.

Definition 1.7. Let $\Omega \subseteq X$ be an open set and let $E \in \mathcal{B}X$. We say that E has (locally) finite perimeter in Ω if $\mathbf{1}_E \in BV(\Omega, \mu)$ $(BV_{loc}(\Omega, \mu))$.

Now we proove, following [41], the coarea formula. The proof is analogous to the Euclidean case.

Proposition 1.1.6 (Coarea formula)**.** *For any $u \in L^1_{loc}(\Omega, \mu)$, if we set $E_t = \{u > t\}$ we have :*

$$\int_{-\infty}^{+\infty} |D\mathbf{1}_{E_t}|(A)dt = |Du|(A) \tag{1.11}$$

for any open set A. In particular, if $u \in BV(\Omega, \mu)$, then for almost every $t \in \mathbb{R}$ the set E_t has finite perimeter and the previous formula holds for every Borel set.

Proof. Given $u \in Lip_{loc}(\Omega)$, $\nabla u \in UG(u)$ and $A \Subset \Omega$, we define the function:

$$m(t) = \int_{E_t \cap A} |\nabla u|(x) d\mu(x).$$

The function m is nonincreasing and bounded, hence differentiable at almost every t. Let then t be a differentiability point of m and define the functions $(g_h)_h : \mathbb{R} \to \mathbb{R}$

$$g_h = \begin{cases} 1 & s \leq t, \\ h(t-s)+1 & t < s \leq t + \frac{1}{h}, \\ 0 & s > t + \frac{1}{h}. \end{cases}$$

We define the sequence $v_h(x) = g_h(u(x))$; we have that $v_h \to \mathbf{1}_{E_t}$ in $L^1_{loc}(A, \mu)$:

$$\int_A |v_h(x) - \mathbf{1}_{E_t}| d\mu(x) = \int_{\{t<u\leq t+\frac{1}{h}\}} g_h(u(x)) d\mu(x) \leq \mu(\{t < u \leq t + \frac{1}{h}\}) \to 0.$$

because $\{t < u \leq t + \frac{1}{h}\} \searrow \emptyset$. So we have to prove that the quantities $|Dv_h|(X)$ are bounded. We note that

$$\begin{aligned} \int_A |\nabla v_h(x)| d\mu(x) &\leq h \int_{\{t<u\leq t+\frac{1}{h}\}\cap A} |\nabla u|(x) d\mu(x) \\ &= h(m(t + \frac{1}{h}) - m(t)). \end{aligned} \tag{1.12}$$

Then passing to the limit $h \to \infty$ in Eq.(1.12), we have that

$$|D\mathbf{1}_{E_t}|(A) \leq \limsup_h |Dv_h|(A) \leq m'(t).$$

Integrating the previous relation we get:

$$\int_{-\infty}^{+\infty} |D\mathbf{1}_{E_t}|(A)dt \leq \int_A |\nabla u| d\mu.$$

By approximation and using the lower semi-continuity of the perimeter, we obtain the same inequality for every BV function u (see Evans and Gariepy [18]). To obtain the referse inequality we approximate u by simple functions. Assuming that u takes values in $[-1,1]$, for any fixed $h \in \mathbb{N}$ we consider numbers $t_{j,h} \in ((j-1)/h-1, j/h-1)(j=1,\ldots,2h)$ such that

$$\frac{1}{h}|D\mathbf{1}_{E_{j,h}}|(A) \leq \int_{(j-1)/h-1}^{j/h-1} |D\mathbf{1}_{E_t}|(A)dt,$$

where $E_{j,h} = \{u > t_{j,h}\}$. Then we define the sequence

$$u_h = -1 + \frac{1}{h}\sum_{j=1}^{2h} \mathbf{E_{j,h}}(x).$$

It is clear that

$$|Du_h|(A) \leq \int_0^1 |D\mathbf{1}_{E_t}|(A)dt. \tag{1.13}$$

Now, we only need to prove that $u_h \to u$ in $L^1(A)$. Define the sets $F_{i,j}$ as

$$F_{i,j} = \{t_{i,h} < u \leq t_{i+1,h}\};$$

then

$$E_{j,h} = \bigcup_{i=j}^{2h} F_{i,j},$$

and

$$u_h(x) = -1 + \frac{1}{h}\sum_{j=1}^{2h}\sum_{i=j}^{2h} \mathbf{F_{i,h}}(x) = -1 + \frac{1}{h}\sum_{i=1}^{2h} i\mathbf{F_{i,h}}(x).$$

Since on $F_{i,h}$ we have that $|u - \frac{i}{h} + 1| \leq \frac{1}{h}$,

$$\int_A |u - u_h| d\mu = \sum_{i=1}^{2h} \int_{F_{i,h}} |u - \frac{i}{h} + 1| d\mu \leq \frac{1}{h}\mu(A) \to 0.$$

A modification of the previous proof gives Eq. (1.13) in the general case, where u assumes values in $(-\infty, +\infty)$. $\square$

Remark 4. If $u \in BV(X,\mu)$ is possible to prove a more general relation, i.e. the general Coarea formula

$$\int_{-\infty}^{+\infty}\left(\int_A v(x)d|D\mathbf{1}_{E_t}|(x)\right)dt = \int_A v(x)d|Du|(x),$$

for any measurable function $v : X \to \mathbb{R}$ and $A \in \mathcal{B}(X)$.

Corollary 1.1.7. *Let x_0 be fixed, then for almost every $\rho > 0$ the ball $B_\rho(x_0)$ has finite perimeter.*

Proof. Fix $x_0 \in X$ and $R > 0$. Let $u(x) = d(x_0, x)$, u is locally Lipschitz, thus $|Du|(B_R(x_0)) < +\infty$. Using the Coarea formula we have

$$|Du|(B_R(x_0)) = \int_{-\infty}^{+\infty} |Du_\rho|(B_R(x_0))d\rho. \tag{1.14}$$

Where $u_\rho = \mathbf{1}_{X\setminus B_\rho(x_0)}$ noticing that $|D\mathbf{1}_{X\setminus B_\rho(x_0)}| = |D\mathbf{1}_{B_\rho(x_0)}|$ and by the arbitraryness of R we get that $B_\rho(x_0)$ has finite primeter $\mathcal{L}^1$-a.e. $\rho \in \mathbb{R}_+$. □

1.2 Sets of finite perimeters in Ahlfor spaces

The purpose of this section is to illustrate the results obtained in the paper [1] in the setting of Ahlfor regular spaces, that will be useful in the next chapter when dealing with perimeter measure of set in Carnot groups. Here we give only some definitions and the main results without proofs.

Definition 1.8. A complete metric mesure space (X, d, μ) is an Ahlfor regular space if $\mu : \mathcal{B}(X) \to [0,\infty]$ is a Borel measure satisfying

$$a\rho^k \leq \mu(B_\rho(x)) \leq A\rho^k \quad \forall x \in X, \quad \rho \in (0, diamX) \tag{1.15}$$

for suitable positive constant a, A with $k \geq 1$.

Moreover we assume that exist a constant $C_P \geq 0$ and $\lambda \geq 0$ such that

$$\int_{B_\rho(x)} |u(y) - u_{x,\rho}|d\mu(y) \leq C_P\rho \int_{B_{\lambda\rho}(x)} |\nabla u|(y)d\mu(y) \tag{1.16}$$

whenever $u : X \to \mathbb{R}$ is a locally Lipschitz function and $|\nabla u|$ is an upper gradient of u (see Def. 1.3) .

The *Hausdorff* $\alpha - $ *dimensional measure* in X will be denoted by $\mathcal{H}^\alpha$. Given $E \subset X$ we denote by $m_E(x,\rho)$ the volume $\mu(E \cap B_\rho(x))$ of E in $B_\rho(x)$. Moreover, $\partial^* E$ stands for the *measure theoretical boundary* of E, i.e., $x \in \partial^* E$ if

$$\textit{neither} \quad \lim_{\rho\downarrow 0} \frac{m_E(x,\rho)}{\mu(B_\rho(x))} = 0 \quad \textit{nor} \quad \lim_{\rho\downarrow 0} \frac{m_E^c(x,\rho)}{\mu(B_\rho(x))} = 0. \tag{1.17}$$

Remark 5. By a result of Cheeger [10], under assumptions (1.15) and (1.16) the formula

$$|\nabla u|(x) := \limsup_{\rho\downarrow 0} \frac{1}{\rho} \sup_{y\in B_\rho(s)} |u(y) - u(x)| \tag{1.18}$$

provides a minimal upper gradient of u whenever $u \in Lip_{loc}(X)$.

Let (X, d, μ) a metric measure spaces, and suppose that fulfills (1.15) and (1.16) and E is a set of finite perimeter in X. We will denote the perimeter of E in $A \subset X$, defined in (1.6), by $P(E, A)$.

Lemma 1.2.1. *We have $P(E, B) = 0$ whenever $B \in \mathcal{B}(X)$ is $\mathcal{H}^{k-1} -$ negligeable.*

The following theorems give us a representation of the perimeter meausure respect the *Hausdorff* measure. The doubling property of the perimeter measure will follow by a density estimate. For the proofs we refer to Ambrosio [1].

Theorem 1.2.2. *The measure $P(E, \cdot)$ is concentrated in the set*

$$\Sigma_c := \left\{ x : \limsup_{\rho\downarrow 0} \rho^{-k} min\{m_E(x,\rho), m_{E^c}(x,\rho)\} \geq c \right\} \subset \partial^* E \tag{1.19}$$

with $c > 0$ depending only on (k, a, A, λ, C_I). Moreover $\partial^* E \setminus \Sigma_c$ is $\mathcal{H}^{k-1}$-negligible, $\mathcal{H}^{k-1}(\partial^* E) < \infty$ and

$$P(E, B) = \int_{B\cap\partial^* E} \theta d\mathcal{H}^{k-1} \quad \forall B \in \mathcal{B}(X) \tag{1.20}$$

for some Borel function $\theta : X \to [c', \infty)$, with $c' = (c/C_I)^{(k-1)/k}/\omega_{k-1}$.

Theorem 1.2.3. *The measure $P(E, \cdot)$ satisfies*

$$\infty > \limsup_{\rho\downarrow 0} \frac{P(E, B_\rho(x))}{\rho^{k-1}} \geq \liminf_{\rho\downarrow 0} \frac{P(E, B_\rho(x))}{\rho^{k-1}} > \tau_1 \tag{1.21}$$

$$\liminf_{\rho\downarrow 0} \rho^{-k} min\{m_E(x,\rho), m_{E^c}(x,\rho)\} > \tau_2 \tag{1.22}$$

for $P(E, \cdot)$-a.e. $x \in X$, with $\tau_1, \tau_2 > 0$ depending only on (k, a, A, λ, C_I).

Corollary 1.2.4. *The measure $P(E,\cdot)$ is a.e. asymptotically doubling, i.e.,*

$$\limsup_{\rho\downarrow 0}\frac{P(E,B_{2\rho}(x))}{P(E,B_\rho(x))}<\infty \quad for\ P(E,\cdot)-a.e.\ x\in X. \tag{1.23}$$

1.3 SBV in doubling metric measure spaces

Now we return to consider a doubling metric measure space (X,d,μ), we will analyse some fine property of BV functions on X, and then we will define SBV functions. For the theory of SBV function in the Euclidean case we refer to the book [5].

1.3.1 Decomposition of the perimeter measure

In a general metric setting, it is not possible to define the normal direction and the reduced boundary, only the measure theoretic boundary of $E\subset X$ make sense; moreover since the metric space has only an homogeneus dimension ($d=logC_D$), we cannot use the Hausdorff measure. We have to define another Hausdorff-like measure ([2]). Let $h:\mathcal{B}(X)\to[0,+\infty]$

$$h(\overline{B}_\rho(x))=\frac{\mu(\overline{B}_\rho(x))}{\rho}; \tag{1.24}$$

due to the doubling property of μ follows that h is a doubling function, i.e. $h(\overline{B}_{2\rho})(x)\leq (C_D/2)h(\overline{B}_\rho(x))$ for every $x\in X$ and $\rho>0$. Using the Carathéodory construction, we define the Hausdorff spherical measure $\mathcal{S}^h$ as

$$\mathcal{S}^h=\liminf_{\rho\downarrow 0}\left\{\sum_{i=0}^{\infty}h(B_i): B_i\in\mathcal{B}(X),\ A\subset\bigcup_{i=0}^{\infty}B_i,\ \mathrm{diam}(B_i)\leq\rho\right\}. \tag{1.25}$$

As consequence of the doubling property of h, a Vitali-type covering theorem holds and we have the following density estimates ([2]) :

$$\limsup_{\rho\downarrow 0}\frac{\nu(B_\rho(x)}{h(\overline{B}_\rho(x))}\geq t\ \forall x\in B\Rightarrow \nu(B)\geq t\mathcal{S}^h(B), \tag{1.26}$$

for any locally finite measure ν in X amd $B\in\mathcal{B}(X)$. A representation thorem for the measure $P(E,\cdot)$ holds also in this context and is similar to theorem (1.2.2). For the proof we refer to [2].

Theorem 1.3.1. *Given a set of finite perimeter E, the measure $P(E,\cdot)$ is concentrated on the set $\Sigma_c \subset \partial^* E$ defined by*

$$\Sigma_c := \left\{ x : \limsup_{\rho \downarrow 0} \rho^{-k} min\{m_E(x,\rho), m_{E^c}(x,\rho)\} \geq c \right\}, \tag{1.27}$$

with $c > 0$ depending only on C_D and c_I. Moreover $\mathcal{S}^h(\partial^* E \setminus \Sigma_c) = 0$, $\mathcal{S}^h(\partial^* E) < \infty$ and there exist $\alpha > 0$ and a Borel function $\theta_E : X \to [\alpha, +\infty]$ such that

$$P(E,B) = \int_{B \cap \partial^* E} \theta_E(x) d\mathcal{S}^h(x), \quad \forall B \in \mathcal{B}(X). \tag{1.28}$$

Finally, the perimeter measure is asymptotically doubling, i.e., for $P(E,\cdot)$-a.e. $x \in X$ we have

$$\limsup_{\rho \downarrow 0} \frac{P(E, B_{2\rho}(x))}{P(E, B_\rho(x))} < \infty. \tag{1.29}$$

Theorem 1.3.2 (Upper bound of the density)**.** *Let E be a set of finite perimeter, and let θ_E be the function of Theorem 1.3.1. Then, $\theta_E \leq C_D$, where C_D is the doubling constant for μ.*

Definition 1.9. Let $u : X \to \mathbb{R}$ be a measurable function and $x \in X$; we define the upper and lower approximate limits of u at x respectively by

$$u^\vee(x) = inf \left\{ t \in \overline{\mathbb{R}} : \lim_{\rho \downarrow 0} \frac{\mu(\{u > t\} \cap B_\rho(x))}{\mu(B_\rho(x)} = 0 \right\}$$
$$u^\wedge(x) = sup \left\{ t \in \overline{\mathbb{R}} : \lim_{\rho \downarrow 0} \frac{\mu(\{u < t\} \cap B_\rho(x))}{\mu(B_\rho(x)} = 0 \right\}.$$

If $u^\vee(x) = u^\wedge(x)$ we call their common value, denoted by $\tilde{u}(x)$, the approximate limit of u at x. We also set $S_u = \{x \mid u^\vee(x) > u^\wedge(x)\}$, the discontinuity set of u.

When $u = \mathbf{1}_E$, then $S_u = \partial^* E$. If $u \in L^\infty_{loc}(X)$ and $x \notin S_u$, then

$$\lim_{\rho \downarrow 0} \frac{1}{\mu(B_\rho(x))} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)| d\mu(y) = 0.$$

We have the following useful characterization of S_u :

Proposition 1.3.3. *Let $u \in L^1(X,\mu)$ then*

$$S_u = \bigcup_{t,s\in D, s\neq t} \partial^*\{u > t\} \cap \partial^*\{u > s\}, \tag{1.30}$$

where $D \subset \mathbb{R}$ is any dense set. In particular if $u \in BV(X)$ we can chose D s.t. for every $s \in D$ the set $\{u > s\}$ has finite perimeter.

Proof. Following [1] we have

$$x \in \partial^*\{u > t\} \Rightarrow t \in [u^\wedge(x), u^\vee(x)]\,. \tag{1.31}$$

If $x \in \partial^*\{u > t\}$, we have $0 < \Theta_*(\{u > t\}, x) \leq \Theta^*(\{u > t\}, x) < 1$. Where $\Theta_*(E,x), \Theta^*(E,x)$ are respectively the lower and the upper densities of E in x. By definition of and $u^\wedge$, $u^\wedge(x) \leq t \leq u^\vee(x)$. Moreover we have

$$x \in S_u \quad \textit{and}\ t \in]u^\wedge(x), u^\vee(x)[\Rightarrow x \in \partial^*\{u > t\}.$$

The condition $t > u^\wedge(x)$ implies that $\Theta^*(\{u > t\}, x) > 0$ and the condition $t < u^\vee(x)$ implies $\Theta_*(\{u > t\}, x) > 0$, so that $x \in \partial^*\{u > t\}$. Now, if $x \in S_u$ then $x \in \partial^*\{u > t\} \cap \partial^*\{u > s\}$ for all $s, t \in]u^\wedge(x), u^\vee(x)[$ then

$$x \in \bigcup_{t,s\in D\ \ s\neq t} \partial^*\{u > t\} \cap \partial^*\{u > s\}.$$

Conversely, if there exist $s < t \in \mathbb{R}$ such that

$$x \in \partial^*\{u > t\} \cap \partial^*\{u > s\}$$

then from (1.31), $u^\wedge(x) \leq s < t \leq u^\vee(x)$ whence $u^\wedge(x) < u^\vee(x)$ and $x \in S_u$. If in addition we have that $u \in BV(X)$, then by the coarea formula we get that almost every set $\{u > t\}$ has finite perimeter, then the choice of D can be done in such a way that, for every $t \in D$ the set $\{u > t\}$ has finite perimeter. □

Now we give a decomposition of the total variation measure of a function $u \in BV(X)$. We split $|Du|$ in three parts: one absolutely continuous with respect to μ, the restriction to S_u, which will be represented in term of $\mathcal{S}^h$, and the Cantor part.

Theorem 1.3.4 ([1]). *Let $u \in BV(\Omega)$; set $|D^d u| = |Du| \llcorner (X \setminus S_u)$ and denote $|Gu|$ the density of $|Du|$ respect to μ. Then, $|D^d u|(B) = 0$ for every $B \in \mathcal{G}(X)$ such that $\mathcal{S}^h(B)$ is finite, and, setting for $x \in S_u$*

$$\theta_u(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)dt, \tag{1.32}$$

we have

$$|Du| = |D^d u| + \theta_u \mathcal{S}^h \llcorner S_u = |Gu|\mu + |D^c u| + \theta_u \mathcal{S}^h \llcorner S_u. \tag{1.33}$$

Proof. For every $B \in \mathcal{B}(X)$, by the coarea formula and by the representation formula for the perimeter we get

$$|Du|(B) = \int_{\mathbb{R}} P(\{u > t\}, B)dt = \int_{\mathbb{R}} \int_{\partial^*\{u>t\} \cup B} \theta_{\{u>t\}}(x)d\mathcal{S}^h(x)dt; \tag{1.34}$$

if $B \subset S_u$, using (1.30), (1.32) and Fubini theorem we get

$$|Du|(B) = \int_{B \cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)dtd\mathcal{S}^h(x). \tag{1.35}$$

If $B \in X \setminus S_u$, then the measure $|Du|$ can be split into two parts, one absolutely continuous with respect to the measure μ whit density $|Gu|$, and one singular with respect to μ; we call this last part the Cantor part of the measure $|Du|$, and then we can write

$$|Du|(B) = \int_B |Gu|d\mu + |D^c u|(B). \tag{1.36}$$

Finally, if $B \cap S_u = \emptyset$ and $\mathcal{S}^h(B) < \infty$, then by (1.30) for every $x \in B$ there is at most one $t \in \mathbb{R}$ such that $x \in \partial^*\{u > t\}$, namely $t = \tilde{u}(x)$. Using (1.34), Fubini theorem and Theorem 1.3.2 we get

$$|Du|(B) \leq C_D \int_B \mathcal{L}^1(\{t \in \mathbb{R} : x \in \partial^*\{u > t\}\})d\mathcal{S}^h(x) = 0. \tag{1.37}$$

□

Following [5] we have the chain rule for BV, that will be useful for a characterization of SBV functions.

$$\Lambda := \{\psi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) : \exists I \textit{ closed interval such that} \tag{1.38}$$

$$\psi'(t) = 0 \ \forall t \notin I, \ \psi \textit{ is strictly increasing in } I\}. \tag{1.39}$$

Theorem 1.3.5. *For every $u \in BV(X)$ and ψ in the class Λ previosly defined, the function $\psi \circ u$ belongs to $BV(X)$ and the following chain rule holds:*

$$|D(\psi \circ u)| = \psi'(\tilde{u})|D^d u| + \Psi(u) S^h \llcorner S_u, \tag{1.40}$$

where

$$\Psi(u)(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(t)\theta_{\{u>t\}}(x)dt. \tag{1.41}$$

Proof. Let $B \subset S_u$, $\psi \in \Lambda$ and $I = [a,b]$. By the coarea forumula, we get, since $\{\psi(u) > t\} = X$ fot $t < \psi(a)$ and $\{\psi(u) > t\} = X$ for $t > \psi(b)$,

$$|D(\psi \circ u)|(B) = \int_{\mathbb{R}} P(\{\psi(u) > t\}, B)dt = \int_{\psi(a)}^{\psi(b)} P(\{\psi(u) > t\}, B)dt. \tag{1.42}$$

If $t \in]\psi(a), \psi(b)[$ and we set $t = \psi(s)$, we get $\{\psi(u) > t\} = \{u > s\}$, and thus

$$\begin{aligned} |D(\psi \circ u)|(B) &= \int_b^a \psi'(s)P(\{u > s\}, B)ds = \int_{\mathbb{R}} \psi'(s)P(\{u > s\}, B)ds & (1.43)\\ &= \int_B \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(s)\theta_{\{u>s\}}(x)dsd\mathcal{S}^h(x). & (1.44) \end{aligned}$$

If $B \subset X \setminus S_u$, then $x \in \partial^*\{u > t\}$ only if $t = \tilde{u}(x)$ (x is an approximate continuity point of u), and then we find

$$\begin{aligned} |D(\psi \circ u)|(B) &= \int_{\mathbb{R}} P(\{\psi(u) > t\}, B)dt = \int_{\mathbb{R}} \psi'(s)P(\{u > s\}, B)ds & (1.45)\\ &= \int_{\mathbb{R}} \int_B \psi'(\tilde{u}(x))dP(\{u > s\}, \cdot)ds = \int_B \psi'(\tilde{u})d|D^d u|. & (1.46) \end{aligned}$$

□

1.3.2 SBV functions

Finally we can define the class of SBV functions in doubling metric mesure space.

Definition 1.10. A function $u \in BV(X)$ is said to be a special function of bounded variation, $u \in SBV(X)$, if the following holds

$$\int_X |Gu|d\mu = \inf\{|Du|(X \setminus K) : K \subset X, \mathcal{S}^h(K) < \infty\}. \tag{1.47}$$

As a direct consequence of the previuos definition and of Theorem 1.3.4 we have the following characterisations of SBV functions.

Proposition 1.3.6. *Given $u \in BV(X)$, $u \in SBV(X)$ if and only if $|D^c u| = 0$.*

There is another characterization of SBV functions based on the chain rule, and it will be the key point for the proof of the closure theorem. For $\psi \in \Lambda$ we set $osc\psi = max\ \psi - min\ \psi$, where Λ is defined in (1.38). For a proof of the following theorem we refer to [2].

Theorem 1.3.7. *Let $u \in BV(X)$; then, u belongs to $SBV(X)$ and $\mathcal{S}^h(S_u) < +\infty$ if and only if*

$$|D(\psi \circ u)| \leq \psi'(\tilde{u})a\mu + osc\psi\nu \tag{1.48}$$

for every $\psi \in \Lambda$. Moreover, given any pair (a, ν), we have

$$a \geq |Gu| \quad \mu - a.e. \quad and \quad osc\psi\nu \geq \Psi(u)\mathcal{S}^h \llcorner S_u \quad \forall \psi \in \Lambda. \tag{1.49}$$

Theorem 1.3.8 ([2])**.** *Let $u \in BV(X)$ and let $(u_n) \subset SBV(x)$ be a sequence convergent to $u \in L^1(X, \mu)$ such that the densities $|Gu_n|$ of the absolutely continuous parts of the measures $|Du_n|$ are bounded in $L^1(X, \mu)$ and equiintegrable, and*

$$\sup_n \mathcal{S}^h(S_{u_n}) < +\infty. \tag{1.50}$$

Then, u belongs to $SBV(X)$ as well.

Proof. By the equiintegrability and boundedness hypotheses, the sequence $(|Gu_n|)$ is weakly compact in $L^1(X, \mu)$, and the sequence $(\mathcal{S}^h \llcorner S_{u_n})$ is weakly* compact, hence we can assume, possibly estracting a subsequence, that $(|Gu_n|)$ weakly converges to some function a in $L^1(X, \mu)$ and that $(\mathcal{S}^h \llcorner S_{u_n})$ weakly* converges in X to some finite positive measure ν. In order to conclude, it suffices to check (see 1.48), hence we start by fixing $\psi \in \Lambda$. As a first step, from the strong convergence of $\psi \circ u_n$ to $\psi \circ u$ in $L^1(X, \mu)$ we deduce that $\psi'(u_n)|Gu_n|$ converges to $\psi'(u)a$ weakly in L^1. In fact,

$$\psi(u_n)|Gu_n| = [(\psi'(u_n) - \psi'(u))|Gu_n|] + \psi'(u)|Gu_n| \tag{1.51}$$

and notice that by Vitali dominated convergence theorem (see [5]) the terms between square brackets tend to 0 in the L^1 norm. Therefore

$$\lim_{n\to+\infty}\int_X \varphi\psi'(u_n)|Gu_n|d\mu = \lim_{n\to+\infty}\int_X \varphi\psi'(u)|Gu_n|d\mu = \int_X \varphi\psi'(u)ad\mu$$

for any $\varphi \in L^\infty(X,\mu)$. The right-hand side of

$$|D(\psi\circ u_n)| \le \psi'(\tilde{u}_n)|Gu_n|\mu + C_D osc\psi\mathcal{S}^h \llcorner S_{u_n} \tag{1.52}$$

weakly* converges to $\psi'(\tilde{u}_n)a\mu + C_D osc\psi\nu$. Fix some sets $A, A' \subset X$ with $A' \subset\subset A$; by the lower semicontinuity of the total variation with respect to the strong convergence in $L^1(X,\mu)$ we have

$$\begin{aligned} |D(\psi\circ u)(A') &\le \liminf_{n\to+\infty} |D(\psi\circ u_n)|(A') \\ &\le \lim_{n\to+\infty}\int_{A'} \psi'(\tilde{u}_n)|Gu_n|d\mu + osc\psi\int_{\bar{A}'\cap S_{u_n}} \theta_{u_n} d\mathcal{S}^h \\ &\le \int_A \psi'(\tilde{u})ad\mu + osc\psi\nu(A). \end{aligned}$$

Taking the supremum among all $A' \subset\subset A$, we obtain

$$|D(\psi\circ u)|(A) \le \int_A \psi'(\tilde{u})ad\mu + osc\psi\nu(A) \tag{1.53}$$

for every open set $A \subset X$, and the proof is complete since A is arbitrary. □

Theorem 1.3.9 (Compactness in SBV(X))**.** *Let $(u_n)_n \subset SBV(X)$ be a sequence such that:*

1. *the sequence $(u_n)_n$ is bounded in BV;*
2. *the functions $|Gu_n|$ are equiintegrable;*
3. *there exist a constant $C > 0$ such that*

$$\sup_{n\in\mathbb{N}} \mathcal{S}^h(S_{u_n}) \le C. \tag{1.54}$$

Then u_n has limit points in the L^1_{loc} topology and any limit point belongs to $SBV(X)$.

Proof. If the sequence $(u_n)_n$ is bounded in BV, we know that up to subsequences it converges in L^1_{loc} to a function $u \in BV(X)$; but then we can apply the closure theorem to conclude that $u \in SBV(X)$. $\square$

Definition 1.11. We say that a metric measure space (X, d, μ) is *local*, or that it is a $\mathcal{U}$-space, if for every pair of finite perimeter sets E and F with $E \subset F$ the equality $\theta_E = \theta_F$ holds $\mathcal{S}^h$-a.e.in $\partial^* E \cap \partial^* F$.

In a $\mathcal{U}$-space, it is possible to obtain a nicer formula for the jump part of the derivative $|Du|$, i.e. the part concentrated on S_u.

Proposition 1.3.10. *Let X be a $\mathcal{U}$-space and let $u \in BV(X)$ with $\mathcal{S}^h(S_u) < \infty$. Then, there is a function $\Theta_u : S_u \to [\alpha, C_D]$ such that*

$$\Psi(u) = [\psi(u^\vee) - \psi(u^\wedge)]\Theta_u, \tag{1.55}$$

for every $\psi \in \Lambda$, where α is the constant in Theorem 1.3.1 and Ψ is defined in (1.41).

Proof. Let as set $\theta_t = \theta_{\{u>t\}}$, let D be a countable dense set in $\mathbb{R}$ with $P(\{u > s\}) < +\infty$ for every $s \in D$, and recall that

$$S_u = \bigcup_{t,s\in D, s\neq t} \partial^*\{u > t\} \cap \partial^*\{u > s\}. \tag{1.56}$$

For every $t \in \mathbb{R}$ define the sets

$$N_t = \bigcup_{s\in D} \{x \in S_u \cap \partial^*\{u > t\} \cap \partial^*\{u > s\} : \ \theta_t(x) \neq \theta_s(x)\}. \tag{1.57}$$

that are $\mathcal{S}^h$-negligible for $\mathcal{L}^1$-a.e. t because for every $s \in D$ the densities θ_s and θ_t coincides S^h-a.e. in $\partial^*\{u > t\} \cap \partial^*\{u > s\}$. In particular, setting $N = \cup_{t\in D} N_t$, we may define a density function Θ_u on S_u such that $\Theta_u = \theta_s$ in $(S_u \cap \partial^*\{u > s\}) \setminus N$ for every $s \in D$. Set

$$\mathcal{N} = \{(x, t) \in S_u \times \mathbb{R} : \ x \in \partial^*\{u > t\}, \Theta_u \neq \theta_t(x)\} \tag{1.58}$$

and notice that for every $t \in \mathbb{R}$ the section $\mathcal{N}_t = \{x \in S_u : \ (x, t) \in \mathcal{N}\}$ coincides (up to a $\mathcal{S}^h$-negligible set) with the set N_t, hence $\mathcal{S}^h(\mathcal{N}_t) = 0$. By Fubini theorem, we have

$\mathcal{L}^1(\mathcal{N}_x) = 0$ for $\mathcal{S}^h$-a.e. $x \in S_u$ where $\mathcal{N}_x = \{t \in \mathbb{R} : \ (x,t) \in \mathcal{N}\}$. Therefore,

$$\Psi(u)(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(t)\theta_t(x)dt \tag{1.59}$$

$$= \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(t)\Theta_u(x)dt \tag{1.60}$$

$$= [\psi(u^\vee(x)) - \psi(u^\wedge(x))]\Theta_u(x) \tag{1.61}$$

$\mathcal{S}^h$-a.e. in S_u for every $\psi \in \Lambda$, and the thesis follows. $\square$

Chapter 2

Sub-Riemannian Manifolds and Tangent Spaces

This chapter is devoted to the study of sub-Riemannian metrics, in particular we discuss the connection between sub-Riemannian geometry and nilpotent groups. Since tangent spaces of a sub-Riemannian manifold are themselves sub-Riemannian manifolds, we can define on them a metric, using Gromov's definition of tangent space of a metric space, and they come with a structure of nilpotent Lie groups with dilations. These kind of problems were first studied by Guy Métiver, Elias Stein and others in the field of hypoelliptic differential equations ([26], [40], [47]).

Sub-Riemannian metrics appear, in the literature, under a variety of names: singular Riemannian metrics ([32]), Carnot-Carathéodory (CC) metrics ([27], [44]), nonholonomic Riemannian metrics ([53]).

For a more detailed discussion we refer to [7], [36] from wich we take most of the material of this chapter. In section (2.1) we give the definitions of Lie group and Lie algebra with some basic properties and we prove, for completeness, a well known results : the exponential map of a connected, simply connected nilpotent Lie group $\mathbb{G}$ is a diffeomorphism between $\mathbb{G}$ and its Lie algebra $\mathfrak{g}$.
In section (2.2) we define Carnot-Carathéodory distance and the CC-spaces. Then we state the Theorem of Chow which implies the finiteness of the CC-distance for every CC-spaces that satisfies the so-called "Chow condition", moreover we state a more precise result of Sussmann [52]. Here we give two examples of CC-spaces that satisfies the "Chow

condition", the Guršin plane G_2 and the more famous Heisenberg group $\mathbb{H}^1$. In the last part of the section we define regular points, priviledged coordinates and by a scaling argument the tangent set to a CC-space, finally we state the main theorem on the structure of the tangent set at regular points.
The third section is idependet from the other two, here it is proved the existence of a minimal upper gradient for CC-spaces.

2.1 Lie groups and algebras

In this section we recall some basic definition of Lie algebras and Lie groups theory, and we state some fundamental properties, a more complete description can be found in [34].

Definition 2.1. A Lie group $\mathbb{G}$ is a manifold endowed with the structure of differential group, i.e. a group where the maps

$$\mathbb{G} \ni x \mapsto x^{-1} \in \mathbb{G} \qquad \mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto xy \in \mathbb{G}$$

are C^∞.

Definition 2.2. A vector space $\mathfrak{g}$ is a Lie algebra if there is a bilinear and antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi's identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \qquad \forall X, Y, Z \in \mathfrak{g}.$$

Given two subalgebras $\mathfrak{h}, \mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ we denote by $[\mathfrak{h}, \mathfrak{m}]$ the vector subspace generated by the elements of $\{[Z, X] : Z \in \mathfrak{h},\ X \in \mathfrak{m}\}$.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra, we define recursively

$$\mathfrak{g}_0 = \mathfrak{g} \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

Then the decreasing sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$

is called the lower central series for $\mathfrak{g}$. We say that $\mathfrak{g}$ is nilpotent if $\mathfrak{g}_j = 0$ for some j.

Definition 2.4. A Lie Group $\mathbb{G}$ is said to be nilpotent if it is connected and if its Lie algebra is nilpotent.

Definition 2.5. Let $\mathbb{R}$ denote the simply connected Lie group of additive reals with 1-dimensional abelian Lie algebra $\mathfrak{r}$ generated by $\left(\frac{d}{dt}\right)_0$, and let $\mathbb{G}$ be a Lie group with Lie albegra $\mathfrak{g}$. If X is given in $\mathfrak{g}$, we can define a Lie algebra homomorphism $\phi : \mathfrak{r} \to \mathfrak{g}$ by requiring that $\phi(t\left(\frac{d}{dt}\right)_0) = tX$.

The corresponding homomorphism $\mathbb{R} \to \mathbb{G}$ is written $t \mapsto \exp tX$. Write $c(t) = \exp tX$, let $\frac{d}{dt}$ and $\tilde{X}$ be the left-invariant vector fields on $\mathbb{R}$ and $\mathbb{G}$, respectively, that extends $\left(\frac{d}{dt}\right)_0$ and X. We have

$$(dc)_t \left(\frac{d}{dt}\right) = \tilde{X}_{c(t)} \quad \textit{and } c(0) = e_G. \tag{2.1}$$

The previous equation says that $c(t) = \exp tX$ is the integral curve for $\tilde{X}$ with $c(0) = e_G$. The equation (2.1), when written in local coordinates, yields a system of ordinary differential equations satisfied by the integral curve in question. From this system of differential equations, one sees that the map of the Lie algebra $\mathfrak{g}$ into $\mathbb{G}$ given by $X \mapsto \exp X$ is smooth. This is the exponential map for $\mathbb{G}$.

Remark 6. If $\Phi : \mathbb{G} \to H$ is a smooth homomorphism, then Φ, the differential $d\Phi_e$ and the exponential map are connected by the formula

$$exp_H \circ d\Phi_e = \Phi \circ exp_{\mathbb{G}}. \tag{2.2}$$

For nilpotent Lie groups we have the following useful theorem, with it we can identify the Lie group with its Lie algebra and then with $\mathbb{R}^n$ (where n is the dimension of the algebra).

Theorem 2.1.1 ([34]). *If $\mathbb{G}$ is a simply connected nilpotent analytic group with Lie algebra $\mathfrak{g}$, then the exponential map is a diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$.*

Proof. The first step is to prove that the exponential map is one-one onto. We proceed by induction on the dimension of the group in question. The trivial case of the induction is dimension 1, where the group in question is $\mathbb{R}$ and the result is known. We begin to coordinatize the group $\mathbb{G}$. Namely we form a decreasing sequence of subalgebras

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_n = 0,$$

with $dim\mathfrak{g}_i/\mathfrak{g}_{i+1} = 1$ and with each $\mathfrak{g}_i$ an ideal in $\mathfrak{g}$. The corresponding analytic subgroups are closed and simply connected, and we are interested in the analytic subgroup Z corresponding to $\mathfrak{z} = \mathfrak{g}_{n-1}$. Note that Z is contained in the center of $\mathbb{G}$, and therefore $\mathfrak{z}$ is contained in the center of $\mathfrak{g}$. Since Z is central, it is normal, and we can form the quotient homomorphis $\varphi : G \to \mathbb{G}/Z$. The group $\mathbb{G}/Z$ is a connected nilpotent Lie group with Lie algebra $\mathfrak{g}/\mathfrak{z}$, and $\mathbb{G}/Z$ is simply connected since Z is conneced and $\mathbb{G}$ is simply connected. The inductive hypostesis is thus applicable to $\mathbb{G}/Z$.
We can now derive our conclusions inductively about $\mathbb{G}$. First we prove the "one-one" property. Let X and X' be in $\mathfrak{g}$ with $exp_{\mathbb{G}}X = exp_{\mathbb{G}}X'$. Application of φ gives $exp_{\mathbb{G}/Z}(X + \mathfrak{z}) = exp_{\mathbb{G}/Z}(X' + \mathfrak{z})$. By inductive hipotesis tor $\mathbb{G}/Z$, $X + \mathfrak{z} = X' + \mathfrak{z}$. Thus $X - X'$ is in the center and commutes with X'. Consequently

$$exp_{\mathbb{G}}X = exp_{\mathbb{G}}(X' + (X - X')) = (exp_{\mathbb{G}}X')(exp_{\mathbb{G}}(X - X')),$$

and we conclude that $exp_{\mathbb{G}}(X - X') = 1$. Since Z is simply connected, the result for dimension 1 implies that $X - X' = 0$. Hence $X = X'$, and the exponential map is one-one for $\mathbb{G}$.
Next we prove the "onto" property. Let $x \in \mathbb{G}$ be given, and choose $X + \mathfrak{z}$ in $\mathfrak{g}/\mathfrak{z}$ with $exp_{\mathbb{G}/Z}(X + \mathfrak{z}) = \varphi(x)$. Put $x' = exp_{\mathbb{G}}X$, then using (2.2) we have

$$\varphi(x') = \varphi(exp_{\mathbb{G}}X) = exp_{\mathbb{G}/Z}(X + \mathfrak{z}) = \varphi(x)$$

so that $x = x'z$ with z in $ker\varphi = Z$. Since Z is connected and abelian, we can find X'' in its Lie algebra $\mathfrak{z}$ with $exp_{\mathbb{G}}X'' = z$. Since X and X'' commute,

$$x = x'z = (exp_{\mathbb{G}}X)(exp_{\mathbb{G}}X'') = exp_{\mathbb{G}}(X + X'').$$

Thus the esponential map is onto on $\mathbb{G}$.
To complete the proof of the theorem, we have to show that the exponential map is everywhere regular. We now fully coordinatize the group $\mathbb{G}$ in question. Let X_i be in $\mathfrak{g}_{i-1}$ but not in $\mathfrak{g}_i$. The coordinates formed from the ordered basis $X_1, \ldots, X_n$ exibit $\mathbb{G}$ as diffeomorphic to $\mathbb{R}^n$. In other words we can write

$$exp(x_1X_1 + \cdots + x_nX_n) = exp(y_1(x_1, \ldots, x_n)X_1) \cdots exp(y_n(x_1, \ldots, x_n)X_n), \qquad (2.3)$$

and what needs to be proved is that the matrix $(\partial y_i/\partial x_j)$ is everywhere nonsingular.
This non singularity will be an immediate consequence of the forumula

$$y_i(x_1, \ldots, x_n) = x_i + \tilde{y}_i(x_1, \ldots, x_{i-1}) \quad for\ i \leq n. \qquad (2.4)$$

To prove the previous equation, we argue by induction on $n = dim\mathbb{G}$. The theorem is trivial for the case $n = 1$. For the inductive case let $\mathbb{G}$ be given, and define $Z, \mathfrak{z}$, and φ as erlier. In term of our basis $X_1, \dots, X_n$, the Lie algebra $\mathfrak{z}$ is given by $\mathfrak{z} = \mathbb{R}X_n$. If we write $d\varphi$ for the differential at e of the homomorphism φ, then $d\varphi(X_1), \dots, d\varphi(X_{n-1})$ is a basis of the Lie algebra $\mathbb{G}/Z$.
Let us apply φ to both sides of (2.3), then (2.2) gives

$$\begin{aligned} exp(x_1 d\varphi(X_1) + \cdots + x_{n-1} d\varphi(X_{n-1})) &= exp(y_1(x_1, \dots, x_n) d\varphi(X_1)) \cdots \\ & \quad exp(y_{n-1}(x_1, \dots, x_n) d\varphi(X_{n-1})). \end{aligned}$$

The left side is independent of x_n, and therefore

$$y_1(x_1, \dots, x_n), \dots, y_{n-1}(x_1, \dots, x_n)$$

are all independent of x_n. We can regard them as functions of $n-1$ variables, and our inductive hypothesis says that they are of the form

$$y_i(x_1, \dots, x_n) = x_i + \tilde{y}_i(x_1, \dots, x_{i-1}) \quad for\, i \leq n-1.$$

This proves (2.4) for all i but $i = n$. Thus let us define $\tilde{y}_n$ by $y_n(x_1, \dots, x_n) = x_n + \tilde{y}_n$. Then we have

$$exp(y_n(x_1, \dots, x_n) X_n) = exp(\tilde{y}_n(x_1, \dots, x_n) X_n) exp(x_n X_n).$$

Since X_n is central, we have also

$$exp(x_1 X_1 + \cdots + x_n X_n) = exp(x_1 X_1 + \cdots + x_{n-1} X_{n-1}) exp(x_n X_n).$$

Using the preceding equations and canceling $exp(x_n X_n)$ from both sides, we obtain

$$\begin{aligned} exp(x_1 X_1 + \cdots + x_n X_n) = exp((x_1 + \tilde{y}_1) X_1) \times \cdots \\ \cdots \times exp((x_{n-1} + \tilde{y}_{n-1}(x_1, \dots, x_{n-2})) X_{n-1}) exp(\tilde{y}_n(x_1, \dots, x_n) X_n). \end{aligned}$$

The left side is independent of x_n, and hence so is the right side. Therefore $\tilde{y}_n$ is independent of x_n, and the proof of (2.4) for $i = n$ is complete. $\square$

Now, we state the celebrated Baker-Campbell-Hausdorff formula, in the sequel we will write BCH, for a proof we refer to [34]. Consider a Lie group $\mathbb{G}$ and its Lie algebre $\mathfrak{g}$

and an element $X \in \mathfrak{g}$, the adjoint operator $adX : \mathfrak{g} \to \mathfrak{g}$ is the linear operator defined as follows

$$adX(Y) := [X, Y] \quad Y \in \mathfrak{g}.$$

Theorem 2.1.2 (BCH formula)**.** *Let $X, Y \in \mathfrak{g}$, where $\mathfrak{g}$ is a nilpotent Lie algebra of a simply connected group $\mathbb{G}$ of step s. Define $X \cdot Y := ln(exp(X)exp(Y))$, then we have*

$$X \cdot Y = \sum_{n=1}^{s} \frac{(-1)^{n+1}}{n} \sum_{1 \le |\alpha|+|\beta| \le s} \frac{(adX)^{\alpha_1}(adY)^{\beta_1} \cdots (adX)^{\alpha_n}(adY)^{\beta_n - 1}Y}{\alpha!\beta!|\alpha+\beta|}$$

for any $\alpha \in \mathbb{N}^n$, where $\alpha! = \prod_1^n \alpha_i$ and $|\alpha| = \sum_1^n \alpha_i$.

2.2 Sub-Riemannian manifolds

Throught this section, M will be a smooth manifold of dimension n. For a more precise discussion of the following theory we refer to [7] and [36], where we took most of the material.

Definition 2.6. An horizontal subbundle of the tangent bundle TM is a distribution of subspaces $H_pM \subset T_pM$, for any $p \in M$, these subspaces are locally generated by Lipschitz vector fields.

Definition 2.7. An horizontal curve is an absolutly continuous map $\gamma : [a, b] \to M$, such that $\gamma'(t) \in H_{\gamma(t)}M$ for a.e. $t \in [a, b]$.

We say that M is H-connected if any two points of M can by joined by an horizontal curve.

Definition 2.8. A H-connected manifold M is said a Carnot-Carathéodory space.

Now, we introduce the notion of sub-Riemannian manifold and Carnot-Carathéodory distance.

Definition 2.9. Consider a smooth manifold M with an horizontal subbundle HM. A quadratic form g on TM such that its restriction $g_{|_{HM}}$ is Lipschitz regular on HM is called a sub-Riemannian metric on M. We said that (M, HM, g) is a sub-Riemannian manifold.

Define the length of an horizontal curve $\gamma : [a, b] \to M$ with respect to the sub-Riemannian metric g as

$$L_g(\gamma) = \int_a^b \sqrt{g(\gamma(t), \gamma'(t))}\, dt.$$

Definition 2.10. Given a sub-Riemanian manifold (M, HM, g) and two points $p, q \in M$ we define the Carnot-Carathéodory distance (CC) $d(p, q)$ as

$$d(p, q) = \inf_{\gamma} \{L_g(\gamma)\}$$

where γ is an horizontal curve which joins p to q.

Suppose that $(X_1, \ldots, X_m)$ with $m < n$ are Lipschitz, linear indipendent vector fields, that generate in a neighbourhood of $p \in M$ the horizontal boundle HM.

Definition 2.11 ([7])**.** A vector $v \in T_pM$ is a subunit vector if $v = \sum_{i=1}^m a_i X_i(p)$, with $\sum_i (a_i)^2 \leq 1$.

Definition 2.12. A subunit curve $\gamma : [c, c'] \to M$ is an absolutely continuous curve, such that

$$\gamma'(t) = \sum_{i=1}^m c^i X_i(\gamma(t)),$$

with $\sum_i (c^i)^2 \leq 1$.

The following theorem gives an equivalent definition of CC-distance using the vector fields X_i and the notion of subunit curve. In the sequel we will use only this definition of CC-distance.

Theorem 2.2.1 ([37])**.** *Let (M, HM, g) be a sub-Riemannian manifold and assume that $dim(H_pM) = m < n$ for any $p \in M$. Then for any $p, q \in M$ we get*

$$d(p, q) = \inf \{c - c' |\ \gamma \in S_{p,q},\ \gamma : [c', c] \to M\},$$

where d is the CC-distance and $S_{p,q}$ is the family of subunit curves γ that joins p to q.

2.2.1 The Theorem of Chow

In this section we state the classical theorem of Chow, following [7] the theorem will be deduced by a more precise result by Sussmann. Consider the control system on M,

$$x' = \sum_{i=1}^{m} u_i(t) X_i(t).$$

For $p \in M$ and $T > 0$, let $\Omega_{p,T}$ be the space of controlled paths with origin p, parametrized by $[0, T]$ and a control function $u \in U_{p,T}$, where $U_{p,T}$ is an open set containing the origin in $L^1([0, T], \mathbb{R}^m)$.

Definition 2.13. We will denote by $E_{p,T} : U_{p,T} \to M$ the mapping which maps u to $x_u(T)$. We will call $E_{p,T}$, the end-point map.

Consider now the accessible set A_p (the set of points accessible in finite time from p, by a controlled path) is the image of $E_{p,T}$ for a chosen T.

Definition 2.14. An immersed submanifold of a manifold M is a subset A of M, endowed with a manifold structure, such that

1. The inclusion map $i : A \to M$ is an immersion;
2. Any continuous map $f : P \to M$, where P is a manifold, taking its values in A, is already continuous when considered as a map $f : P \to A$, where A is endowed with its manifold topology.

Remark 7. The role of condition (2) in the previous definition is to prevent the existence of curves in A, asymptotic in M to some point $a \in A$, while staying far from a in the own topology of A.

For the proof of the following theorems we refer to [7].

Theorem 2.2.2 (Sussman[52], Stefan[49])**.** *The set A_p of point accessible form a given point p in M is an immersed submanifold.*

Theorem 2.2.3 (Chow's theorem, Chow[12])**.** *Suppose M is connected and the following condition holds:*
(C) The vector fields X_i and their iterated brackets $[X_i, X_j]$,$[[X_k, X_j], X_j]$, etc. span the tangent space T_xM at every point of M.
Then every two point of M are accessible. Condition (C) is called Chow's Condition.

Chow's Condition is also known under the name of Lie Algebra Rank Condition since the "rank" at every point x of the Lie algebra generated by X_i's is full. In the context of PDEs, it is known under the name of Hörmander's Condition: when it holds, the differential operator $X_1^2 + \cdots + X_m^2$ is hypoelliptic. The formulation of Chow's condition in terms of Lie brackets is due to Hörmander.

Theorem 2.2.4 ([7])**.** *Suppose Chow's condition holds. Then the end-point mapping is open.*

As a consequence of the previous theorem we can compare the topology defined by the sub-Riemannian distance and the original one.

Corollary 2.2.5. *Suppose Chow's condition holds. The the topology defined by the sub-Riemannian distance d is the topology of M.*

Proof. Any ball $B^d(p, \epsilon)$, which is a neighbourhood of p, is the image under E_p of the ball $B(0, \epsilon)$ in L^1, therefore it is an open set in M. Conversely, any neighbourhood U of p contains a ball $B^d(p, \epsilon)$: since E_p is continuous at 0, there exist ϵ such that E_p maps $B^{L^1}(0, \epsilon)$ into U. □

Example 6. Our first example is the so called Grušin plane G_2. As underlying manifold we take $\mathbb{R}^2$ and consider the sub-Riemannian metric defined by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Chow's condition holds, in fact these vector fields with the commutator

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

span the tangent space everywhere, and the sub-Riemannian distance d is complete.
If we set $\delta_\lambda(x, y) = (\lambda x, \lambda^2 y)$ we have

$$(\delta_\lambda)^* X_1 = \lambda^{-1} X_1, \qquad (\delta_\lambda)^* X_2 = \lambda^{-1} X_2, \tag{2.5}$$

it follows that the length of a controlled path is multiplied by $|\lambda|$ under the dilation.

Using homogeneity is easy to give a bound to $d\left((0,0),(x,y)\right)$. First note that on the boundary of the square $|x|\le 1,\ |y|\le 1$ we have

$$1 \le d\left((0,0),(x,y)\right) \le 3,$$

and using homogeneity we get the estimates

$$\sup(|x|,|y|^{1/2}) \le d\left((0,0),(x,y)\right) \le 3\sup(|x|,|y|^{1/2}).$$

This means that the balls $B(0,\epsilon)$ are similar to $[-\epsilon,\epsilon]\times[-\epsilon^2,\epsilon^2]$. For balls centered at regular points (see 2.16) we have the classical Riemannian estimate.

Example 7. The second example is the Heisenberg group: consider $\mathbb{R}^3$, with the sub-Riemannian metric generated by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}.$$

This two vector fields with $[X_1,X_2] = X_3$ span $\mathbb{R}^3$ everywhere, so every point can be reached from any other point. We can define a one parameter group of dilations

$$\delta_\lambda : (x,y,z) \to (\lambda x, \lambda y, \lambda^2 z),$$

so as in the previous example one can prove estimates of the form

$$C(|x|+|y|+|z|^{1/2}) \le d(0,(x,y,z)) \le C'(|x|+|y|+|z|^{1/2}). \tag{2.6}$$

2.2.2 Privileged coordinates and the structure of the tangent space

In the sequel, we will fix a manifold M, of dimension n and a system of vector fields $X_1,\dots,X_m$ on M, and a point p in M. We suppose that $X_1,\dots,X_m$ verify Chow's condition. We denote by d the sub-Riemannian distance induced on M by $X_1,\dots,X_m$. Detailed proofs of the following theorems can be found in [7].

Let $\mathcal{L}^1 = \mathcal{L}^1(X_1,\dots,X_m)$ the set of linear combinations of the vector fields $X_1,\dots,X_m$, we define recursively

$$\mathcal{L}^s = \mathcal{L}^{s-1} + \left[\mathcal{L}^1, \mathcal{L}^{s-1}\right].$$

Denote by $L^s(p)$ the subspace of T_pM of the values taken at the point p by the vectors field belonging to $\mathcal{L}^s$.

Definition 2.15. Consider a system of coordinates centered at p such that the differentials $dy_1, \dots, dy_n$ form a basis of T_p^*M adapted to the flag

$$\{0\} = L^0(p) \subset L^1(p) \subset \cdots \subset L^s(p) \subset \cdots \subset L^r(p) = T_pM. \tag{2.7}$$

Such a coordinate system will be said linearly adapted at p.

Definition 2.16. We say that p is a regular point if $n_s(q) = dimL^s(q)$ remains constant for q in a neighbourhood of p.

Note that the structure of the flag (2.7) may be described by two non-decreasing sequences of integers. The fist one is the sequence

$$0 = n_0 \leq n_1 \leq \cdots \leq n_r = n \tag{2.8}$$

of dimensions of the $L^j(p)$'s. Given a basis $y_1, \dots, y_n$ adapted to the flag, the second sequence is

$$w_1 \leq w_2 \leq \cdots \leq w_n \tag{2.9}$$

where one sets $w_j = s$ if y_j belongs to $L^s(p)$ and does not belong to $L^{s-1}(p)$.

Definition 2.17. We shall say that w_j is the weight of coordinate y_j.

Definition 2.18. Given a function smooth $f : M \to \mathbb{R}$ call $X_1f, \dots, X_mf$ the nonholonomic partial derivatives of order 1 of f. One can also define, in the same way, the nonholonomic partial derivatives of order $k > 1$, i.e. $X_{i_1} \cdots X_{i_k} f$.

Proposition 2.2.6. *Let s be a non-negative integer. For a smooth function f defined near p, the following conditions are equivalent:*

1. *One has $f(q) = O(d(p,q)^s)$ for q near p.*

2. *The nonholonomic derivatives of order $\leq s-1$ of f all vanish at p.*

When these conditions hold we say that f is of order $\geq s$ at p. We say that f is of order s at p if it is of order $\geq s$, and not of order $\geq s+1$.

Definition 2.19. We call system of privileged coordinates a system of local coordinates $z_1, \dots, z_n$ centered at p such that:

1. $z_1, \dots, z_n$ are linearly adapted at p;

2. The order of z_j at p is w_j.

Remark 8. The existence of privileged coordinate for any M is proved in [7], Section 4.3. Moreover, one can prove that canonical coordinates of the first and second kind, in Lie group theory, are privileged coordinates.

Remark 9. The motivation for introducing privileged coordinates is the following estimate that is generically false, as soon as $r \geq 3$, for linearly adapted coordinates :

$$d(0, (y_1, \dots, y_n)) \asymp |y_1|^{1/w_1} + \cdots + |y_n|^{1/w_n}.$$

Following [7] we define the tangent space to M at p and we will see that at regular points the tangent space has a group structure.

Definition 2.20. A differential operator P is said to have order $\geq k$ at p if Pf ha order $\geq k + s$ at p whenever f has order $\geq s$. It has order k at p if it has order $\geq k$ but not $\geq k + 1$.

Let $z_1, \dots, z_n$ a system of privileged coordinates, define the 1-parameter group of dilations

$$\delta_\lambda \ : \ (z_1, \dots, z_n) \mapsto (\lambda^{\omega_1} z_1, \dots, \lambda^{\omega_n} z_n).$$

So relative to the chosen system of privileged coordinates, we have a notion of homogeneity: f is homogeneus of degree s if

$$f(\delta_\lambda z) = \lambda^s f(z).$$

In privileged coordinates, using Taylor expansion we can compute the order of a vector field X at p, only assigning to ∂_{z_j} the weight $-w_j$.

Relative to a privileged coordinate system we have a notion of homogeneous differential operator: P is weighted homogeneous of degree s if

$$\delta_\lambda^* P = \lambda^s P,$$

where the action of δ_λ on differential operators is given by

$$(\delta_\lambda^* P)(\delta_\lambda^* f) = \delta_\lambda^*(Pf),$$

with $\delta_\lambda^* f = f \circ \delta_\lambda$.
Noticing that the defining vector fields X_i have order ≥ -1 at p, they can be expanded in a series of homogeneous vector fields :

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \cdots$$

where $X_i^{(k)}$ has degree k, we set $\widehat{X}_i = X_i^{(-1)}$.

Definition 2.21. We call the system of vector fields $(\widehat{X}_1, \ldots, \widehat{X}_m)$ the canonical nilpotent homogeneous approximation of the system $(X_1, \ldots, X_m)$.

Consider now $\mathbb{R}^n$ with the sub-Riemannian distance $\widehat{d}$ defined from $\widehat{X}_1, \ldots, \widehat{X}_m$. Since the vector fields $\widehat{X}_i$ are homogeneous of degree -1, we have the homogeneity of the distance :

$$\widehat{d}(\delta_\lambda x, \delta_\lambda y) = \lambda \widehat{d}(x, y).$$

Proposition 2.2.7 ([7])**.** *The vector fields $\widehat{X}_i$, generate a nilpotent Lie algebra of step $r = w_n$, $Lie(\widehat{X}_1, \ldots, \widehat{X}_m)$. They satisfy Chow's condition at every point $x \in \mathbb{R}^n$, and the distance $\widehat{d}(x, y)$ is finite for every $x, y \in \mathbb{R}^n$.*

Definition 2.22. We will call $Lie(\widehat{X}_1, \ldots, \widehat{X}_m)$ the tangent Lie algebra, at the point p, of $Lie(X_1, \ldots, X_m)$.

Remark 10. Let $z_1', \ldots, z_n'$ be another system of privileged coordinates around p and let $Lie(\widehat{X}_1', \ldots, \widehat{X}_m')$ be the corresponding tangent Lie algebra, it is possible to prove that there is an isomorphism between $Lie(\widehat{X}_1, \ldots, \widehat{X}_m)$ and $Lie(\widehat{X}_1', \ldots, \widehat{X}_m')$.

Definition 2.23. The space $\mathbb{R}^n$ endowed with the sub-Riemannian structure defined by the vector fields $\widehat{X}_1, \ldots, \widehat{X}_m$ is called the tangent space of M at p.

Denote by G_p the group generated by the diffeomorphisms $exp(t\widehat{X}_i)$ acting on $\mathbb{R}^n$. Since the tangent Lie algebra is nilpotent, by Theorem 2.1.1, G_p is symply connected Lie group having $\mathfrak{g}_p = Lie(\widehat{X}_1, \ldots, \widehat{X}_m)$ has its Lie algebra.

The action of G_p on $\mathbb{R}^n$ is transitive by Chow's theorem, assigning pg to g we can define the map

$$\Psi_p : G_p \to \mathbb{R}^n, \tag{2.10}$$

that maps the identity of G_p to 0. Denoting by H_p the isotropy subgroup of p in G_p we get the bijection

$$\varphi_p : G_p/H_p \to \mathbb{R}^n. \tag{2.11}$$

Example 8. Recall the Grušin system

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

At $p = (0,0)$ we can take $z_1 = x$ and $z_2 = y$ as privileged coordinates. The vector fields X_1 and X_2 are homogeneous of degree -1, so a basis of the tangent space at the origin is given by

$$\widehat{Y}_1 = X_1 \qquad \widehat{Y}_2 = [X_1, X_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To obtain all of $\mathfrak{g}_p$, we need to add $\widehat{Y}_3 = X_2$, so $\mathfrak{g}_p$ is isomorphic to the Heisenberg Lie algebra, $\mathfrak{h}_p = \mathbb{R}X_2$. Thus $G_2 = H_3/\exp(\mathbb{R}X_2)$, where H_3 is the Heisenberg group.

We have proved that the tangent space at p is an homogeneous space but if p is a regular point (see 2.16) we get a richer structure :

Proposition 2.2.8. *If p is a regular point, then $H_p = \{0\}$, and T_pM is isometric to the group $G_p = exp(\mathfrak{g}_p)$.*

2.2.3 Gromov's notion of tangent space

We recall the notion of Hausdorff distance between two subset of $\mathbb{R}^n$ and then we generalize it to metric spaces. We have $H - dist(A, B) \leq \rho$ if any point of A is within distance ρ of B, and any point of B is within distance ρ of A. We say that A_n converges to A if for every compact set K in $\mathbb{R}^n$, we have

$$\lim_{n\to\infty} H - dist(A_n \cap K, A \cap K) = 0. \tag{2.12}$$

We define the tangent cone to S, a closed subset of $\mathbb{R}^n$, at $p \in S$ by

$$T_pS = \lim_{\lambda\to\infty} \delta_{p\lambda} S, \tag{2.13}$$

provided the limit exist. Here $\delta_{p\lambda}$ is the dilation of center p and ratio λ.

Gromov [27] extended this definitions to arbitrary metric spaces. The dilation λM is the metric spaces with all distances multiplied by λ while the Hausdorff distance between two metric spaces X and Y is defined as the infimum of real numbers ρ for which there exist isometric embeddings of X and Y in a metric space Z, $i : X \to Z$ and $j : Y \to Z$, such that $H - dist(i(X), j(Y)) \leq \rho$.

Definition 2.24. A sequence of pointed metric spaces (X_n, x_n) is said to converge to (X, x) if

$$\lim_{n\to\infty} H - dist(B^{X_n}(x_n, R), B^X(x, R)) = 0 \tag{2.14}$$

for any positive R.

Definition 2.25. The tangent space to M at p is defined by

$$(T_pM, 0) = \lim_{\lambda\to\infty} (\lambda M, p), \tag{2.15}$$

provided the limit exists.

Remark 11. Multiplying by λ' the tangent space we have

$$\lambda'(T_pM, 0) = \lim_{\lambda\to\infty} (\lambda\lambda' M, p) = (T_pM, 0),$$

so it possesses a 1-parameter group of dilations having 0 as fixed point. Replacing λ by ϵ^{-1} in (2.15) one obtain

$$\lim_{\epsilon\to 0} \epsilon^{-1} B^M(p, R\epsilon) = B^{T_pM}(0, R).$$

Thus the existence of T_pM means that small balls $B(p, \epsilon)$ in M get more and more similar when $\epsilon \to 0$.
When M is a C^1 Riemannian manifold, one recovers in a metric way the tangent space T_pM with its Euclidean metric.

Finally we have the identification between the metric tangent space and the tangent space in the sub-Riemannian sense:

Theorem 2.2.9 ([7]). *A sub-Riemannian manifold M admits a metric tangent space T_pM, in Gromov sense, at every point p. This space is isometric to the space $\mathbb{R}^n$ endowed with the sub-Riemannian metric associated to the vector fields $\widehat{X}_1, \ldots, \widehat{X}_m$ (see 2.21). At regular points, it has a natural group structure.*

Remark 12 ([7]). Why the tangent space at regular points is a group? The answer seems to be uniformity of convergence. The group structure in T_pM for regular p has a purely metric derivation, we will give a sketched proof of this fact.

Following the work [14], consider the tangent boundle TM of a differentiable manifold M, it has a structure similar to the groupoid $M \times M$. The composition law in $M \times M$ is

$$(p, q) * (p', q') = \begin{cases} (p, q') & if\ p' = q, \\ \text{not defined} & if\ p' \neq q. \end{cases}$$

while for TM the law is

$$(p, v) * (p', v') = \begin{cases} (p, v + v') & if\ p' = p, \\ \text{not defined} & if\ p' \neq p. \end{cases}$$

The tangent boundle TM is a union of groups, in [14] is proved that its structure may be derived from that of $M \times M$ by blowing up the diagonal in $M \times M$. With this observation, and reconsidering the metric structure, we should have

$$TM = \lim_{\epsilon \to 0} \epsilon^{-1}(M \times M).$$

After this digression we return to the sub-Riemannian case and we try to obtain the same conclusions. The tangent space at p is a limit of pointed spaces

$$(T_pM, 0) = \lim_{\epsilon \to 0} \epsilon^{-1}(M, p), \tag{2.16}$$

and it is an homogeneus space, i.e. a metric space with a 1-parameter group of dilations. When p is a regular point the convergence is uniform in a neighbourhood of p , we mean that, for any $R > 0$,

$$H - dist(B^{\epsilon^{-1}M}(q, R), B^{T_qM}(q, R))$$

tends to zero as ϵ tends to zero, uniformly with respect to q.

Note that (2.16) is equivalent to the existence of some mapping

$$\phi_q : T_q M \to M$$

such that

$$\widehat{d^q}(X,Y) - d(\phi_q(X), \phi_q(Y)) = o(\epsilon)$$

if $\widehat{d^q}(0,X) \leq \epsilon$ and $\widehat{d^q}(0,Y) \leq \epsilon$, where $\widehat{d^q}$ is the CC distance defined in (2.21). From the regularity of p, follows that $o(\epsilon)$ is uniform with respect to q, in some neighbourhood of p.

In order to define a composition law on T_pM, consider (p,X), $(p,Y) \in T_pM$ and their images under ϕ_p : (p,q) and (p,r) respectively. We would want to use the product defined on $M \times M$ but it is not possible unless Y belongs to T_qM. We can bypass this problem using dilation and define

$$X * Y := \lim_{\lambda \to \infty} \phi_{pq}^{-1} \delta_{q,\lambda^{-1}} \phi_{pq} \delta_{q,\lambda} Y \tag{2.17}$$

where $q = \phi_p(X)$ and ϕ_{pq} denotes $\phi_q^{-1}\phi_p$. One can prove that the limit (2.17) exists and it defines a group structure on T_pM.

2.3 Upper Gradients

In this section we prove that in CC spaces we can find a natural minimal upper gradient of $u \in L^1_{loc}$. Let $\Omega \subset \mathbb{R}^n$ be open and connected and let $X_1, \ldots, X_k$ be vector fields defined in Ω, with real locally Lipschitz continuous coefficients. Given $u \in \mathrm{Lip}(\Omega)$ set $Xu = (X_1 u, \ldots, X_k u)$, and hence

$$|Xu(x)| = \left(\sum_{j=1}^{k} |X_j u(x)|^2 \right)^{1/2}.$$

Definition 2.26. An absolutely continuous curve $\gamma : [0,T] \to \Omega$ is admissible or subunit if there exist measurable functions $c_j(t)$, satisfying $\sum_1^k c_j(t)^2 \leq 1$ and

$$\gamma'(t) = \sum_1^k c_j(t) X_j(\gamma(t)).$$

Proposition 2.3.1. *A mapping $\gamma : [0,T] \to \Omega$ is an admissible curve if and only if it is 1-Lipschitz i.e., $d(\gamma(b),\gamma(a)) \le |b-a|$ for all a,b.*

Proof. If γ is an admissible curve it follows easily that it is 1-Lipschitz. Suppose now that γ is 1-Lipschitz curve, so it is Lipschitz with respect to the Euclidean metric on Ω and hence is differentiable a.e. Let $t_0 \in (0,T)$ be any point of differentiability of γ. Since $d(\gamma(t_0+\epsilon),\gamma(t_0)) \le \epsilon$ for $\epsilon > 0$, for every $\delta > 0$ there exists an admissible curve $\nu : [0,\epsilon+\delta] \to \Omega$, $\nu(0) = \gamma(t_0)$ and $\nu(\epsilon+\delta) = \gamma(t_0+\epsilon)$. Choosing $\delta = o(\epsilon)$, we have

$$\int_0^{\epsilon+\delta} \nu'(t)\,dt = \gamma(t_0+\epsilon) - \gamma(t_0) = \epsilon\,\gamma'(t_0) + o(\epsilon).$$

By the definition of an admissible curve there exist measurable functions $c_j(t)$ such that $\sum_j c_j(t)^2 \le 1$ and

$$\begin{aligned}\nu'(t) &= \sum_j c_j(t) X_j(\gamma(t_0)) + \sum_j c_j(t)\left(X_j(\nu(t)) - X_j(\nu(0))\right)\\ &= \sum_j c_j(t) X(t_0)) + a(t).\end{aligned}$$

Note that $C|\nu(t)-\nu(0)| \le d(\nu(t),\nu(0)) \le t$, provided ϵ is sufficently small. Hence $|a(t)| \le |X(\nu(t)) - X(\nu(0))| \le Ct$, as the vector fields have locally Lipschitz coefficients. Thus we have

$$\begin{aligned}\gamma'(t_0) &= \epsilon^{-1}\int_0^{\epsilon+\delta} \nu'(t)\,dt + \frac{o(\epsilon)}{\epsilon}\\ &= \frac{1}{\epsilon}\sum_j \left(\int_0^{\delta+\epsilon} c_j(t)\,dt\right) X_j(\gamma(t_0)) + \epsilon^{-1}\int_0^{\epsilon+\delta} a(t)\,dt + \frac{o(\epsilon)}{\epsilon}.\end{aligned}$$

Selecting a sequence $\epsilon_l \to 0$ we conclude that

$$\gamma'(t_0) = \sum_j b_j X_j(\gamma(t_0)), \qquad \sum_j b_j^2 \le 1.$$

$\square$

Proposition 2.3.2. *$|Xu(x)|$ is an upper gradient of $u \in C^\infty(\Omega)$ on the space (Ω, d_c).*

Proof. Let $\gamma : [a,b] \to (\Omega, d_c)$ be a 1-Lipschitz curve. Every admissible curve is Lipschitz so we get that $u \circ \gamma$ is Lipschitz and hence

$$|u(\gamma(b)) - u(\gamma(a))| = \left| \int_a^b \langle \nabla u(\gamma(t)), \gamma'(t)\rangle\; dt \right| \le \int_a^b |Xu(\gamma(t))|\,dt.$$

The last inequality follows form the the fact that γ is admissible by Proposition (2.3.1) and from the Schwartz inequality. $\square$

We proved that $|Xu|$ is an upper gradient of $u \in C^\infty(\Omega)$, moreover $|Xu|$ is also the minimal upper gradient for $u \in L^1_{loc}(\Omega)$, as the following theorem shows.

Theorem 2.3.3. *Let $0 \leq g \in L^1_{loc}(\Omega)$ be an upper gradient on (Ω, d_c) of a function u which is continuous with respect to the Euclidean distance. Then the distributional derivatives $X_j u$, $j = 1, \dots, k$, are locally integrable in Ω and $|Xu| \leq g$ a.e..*

Proof. Here we give only the proof in the case $u \in C^\infty(\Omega)$, for the general case we refer to [30]. Since u is smooth, we have only to prove that $g \geq |Xu|$ a.e. Since the set of point where $|Xu| > 0$ is open and the desired inequality holds outside this set, we can assume that $|Xu| > 0$ everywhere in Ω. Let $|a_j(x)| = X_j u(x)/|Xu(x)|$ and let γ be an integral curve of the vector field $Y = \sum_j a_j X_j$: Clearly γ is an admissible curve, thus is 1-Lipschitz and hence

$$|u(\gamma(t_2)) - u(\gamma(t_1))| \leq \int_{t_1}^{t_2} g(\gamma(t))\, dt.$$

On the other hand we have

$$|u(\gamma(t_2)) - u(\gamma(t_1))| =| \int_{t_1}^{t_2} \langle \nabla u(\gamma(t)), \gamma'(t) \rangle \; dt |= \int_{t_1}^{t_2} |Xu(\gamma(t))|\, dt.$$

This yields

$$\int_{t_1}^{t_2} |Xu(\gamma(t))|\, dt \leq \int_{t_1}^{t_2} g(\gamma(t))\, dt. \tag{2.18}$$

If the vector field Y were parallel to one of the coordinate axes, then (2.18) would imply that $g \geq |Xu|$ a.e. on almost every line parallel to that axis and hence $g \geq |Xu|$ a.e. in Ω. The general case can be reduced to the previous one by the rectification theorem (see [6]), i.e. fix a $y_0 \in \Omega$ then there exist a local diffeomorphism $F : U_0 \ni y_0 \to V \subset \mathbb{R}^n$ of class C^1 such that $[DF(x)]\, X_y = e_1$, for all $y \in U_0$, where e_1 is the first vector of the canonical basis of $\mathbb{R}^n$. $\square$

Chapter 3

Carnot groups

In the previous chapter we saw that at regular points p, the structure of the tangent space T_pM consists of a simply connected nilpotent Lie group, where its Lie algebra $\mathfrak{g}$ is graded and generated by its component of degree 1, say $\mathfrak{g}^1$. There is also a 1-parameter group (δ_λ) of group automorphisms of T_pM, obtained from the grading of $\mathfrak{g}$. Moreover we have on T_pM a left-invariant sub-Riemannian metric obtained from a basis of $\mathfrak{g}^1$.
Such a space is called a Carnot group by Pansu in [44] and Gromov in [7]. The structure of a Carnot group is similar to that of a vector space (replacing abelian with "nilpotent") equipped with a Euclidean metric. Neverthless there are many algebraically non isomorphic Carnot groups having the same dimension n, uncountably many for $n \geq 6$, in contrast with the Euclidean case. Carnot groups are an example of non Euclidean space, neverthless they have enough structure to perform analysis on them.

This chapter is organized as follows: in Section (3.1) we define Carnot groups, the homogeneus dilations δ_λ and we consider the CC-distance associated to the vector fields that generate the first layer of the Lie algebra. By Theorem (2.1.1) we can identify a Carnot group $\mathbb{G}$, by exponential coordiantes, with $\mathbb{R}^n$ and the product law in $\mathbb{R}^n$ is given in Proposition (3.1.1). Then we prove that the Hausdorff dimension of $\mathbb{G}$ is Q,the homogeneus dimension of $\mathbb{G}$, moreover we show that the measures $\mathcal{H}^Q$ and $vol_{\mathbb{G}}$ (the Haar measure on $\mathbb{G}$) are scalar multiples of $exp_\sharp \mathcal{L}^n$.
In Section (3.2) we develop the theory of locally finite perimeter sets, following the work [4] we define the X derivative of a $L^1(\mathbb{G})$ functions with respect to a divergence-free vector field and the sets of finite perimeter as sets $E \subset \mathbb{G}$ whose X-derivative with respect to

the vector fields that generate the first layer are Radon measure. Given a set $E \subset \mathbb{G}$, we introduce, as in the Euclidean case, the measure theoretic boundary $\partial^* E$ and the reduced boundary $\mathcal{F}E$. Then we define the tangent set of a locally finite perimeter set, and we give the notion of vertical halfspaces. Finally we state the main theorem on the structure of the tangent spaces of sets, i.e. the existence of a vertical halfspace at $\mathcal{H}^{Q-1}$ a.e. point of $\partial^* E$.

In Section (3.3) we prove the locality property of Carnot groups, see Definition (1.11). The proof will follows using the same methods of [4]: we give a generalized definition of tangent set considering couples of blow-ups of two different sets, then we apply the tecniques of [4] to prove the existence of a couple of vertical halfspaces in this generalized tangent space. Then a local property of the outer normal is proved, and as a consequence, by a differentiation argument, we get the result.

In the last Section (3.4) we define H-regular surfaces, these surfaces play the same role of C^1 surface in Euclidean spaces. At the end of the section we give an intrinsic definition of rectifiability in Carnot groups build on them. This definition is motivated by the pure k-unrectifiability of the Heisenberg group in the classical sense (see [3], [37]), i.e. images of Euclidean sets via Lipschitz map.

3.1 Definitions and basics properties

Definition 3.1. A Carnot group G of step $s \geq 1$ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step s stratification, i.e.

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s \tag{3.1}$$

with $[V_j, V_1] = V_{j+1}$, $1 \leq j \leq s$, $V_{s+1} = 0$. We keep the notation $n = \sum_i dimV_i$ for the topological dimension of G, and we denote by

$$Q := \sum_{i=1}^{s} idimV_i \tag{3.2}$$

the so called homogeneous dimension of G.

Definition 3.2. Consider a family of inhomogeneous dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\delta_\lambda(\sum_{i=1}^{s} v_i) := \sum_{i=1}^{s} \lambda^i v_i \quad \lambda \geq 0 \tag{3.3}$$

where $X = \sum_{i=1}^{s} v_i$ with $v_i \in V_i$, $1 \leq i \leq s$. The dilations δ_λ belong to $GL(\mathfrak{g})$ and are uniquely determined by the homogeneity conditions

$$\delta_\lambda X = \lambda^k X \quad \forall X \in V_k. \tag{3.4}$$

Remark 13. As we have seen in the previous chapter on a Carnot group $\mathbb{G}$, it is possible to define a metric: the Carnot-Caratheodory left invariant distance on $\mathbb{G}$. Denote by m the dimension of V_1, and fix an inner product in V_1 and an orthonormal basis $X_1, \dots, X_m$ of V_1.
We define the CC metric d as

$$d^2(x,y) := \inf \left\{ \int_0^1 \sum_{i=1}^{m} |a_i(t)|^2 dt : \gamma(0) = x,\ \gamma(1) = y \right\}, \tag{3.5}$$

where the infimum is made among all Lipschitz curves $\gamma : [0,1] \to G$ with the property $\gamma'(t) = \sum_{i=1}^{m} a_i(t)(X_i)_{\gamma(t)}$ for a.e. $t \in [0,1]$. This distance is equivalent to the CC distance defined in (2.10) if we set $g(\gamma(t), \gamma'(t)) = \sum_i |a_i|^2$, where g as in Definition 2.9.

Remark 14. A Carnot group G is cleary nilpotent and by Theorem 2.1.1 we have that the exponential map $exp : \mathfrak{g} \to G$ is a diffeomorphism. So any element $g \in G$ can be identified with $exp(X)$ for some $X \in \mathfrak{g}$, and uniquely written in the form

$$exp(\sum_{i=1}^{s} v_i), \quad v_i \in V_i,\ 1 \leq i \leq s, \tag{3.6}$$

With this identification we can define a family of intrinsic dilation $\delta_\lambda : G \to G$, $\lambda \geq 0$ by

$$\delta_\lambda(exp(\sum_{i=1}^{s} v_i)) := exp(\sum_{i=1}^{s} \lambda^i v_i), \tag{3.7}$$

or we can write it more briefly $exp \circ \delta_\lambda = \delta_\lambda \circ exp$.

Given $\lambda, \nu \geq 0$, we have $\delta_\lambda \circ \delta_\nu = \delta_{\lambda\nu}$, and the BCH formula (2.1.2) gives

$$\delta_\lambda(xy) = \delta_\lambda(x)\delta_\lambda(y) \quad \forall x, y \in G. \tag{3.8}$$

The Carnot-Caratheodory distance is well-behaved under these dilations, namely

$$d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x,y) \quad \forall x, y \in G. \tag{3.9}$$

In the sequel we will need another relation between dilations in the group and the algebra :

$$X(u \circ \delta_\lambda)(g) = (\delta_\lambda X)u(\delta_\lambda g) \qquad \forall g \in \mathbb{G},\ \lambda \geq 0. \tag{3.10}$$

Using the definition of δ_λ, we have

$$\begin{aligned} X(u \circ \delta_\lambda)(g) &= \frac{d}{dt} u \circ \delta_\lambda(g\ exp(tX))\Big|_{t=0} = \frac{d}{dt} u(\delta_\lambda g \delta_\lambda\ exp(tX))\Big|_{t=0} \\ &= \frac{d}{dt} u(\delta_\lambda g\ exp(t\delta_\lambda X))\Big|_{t=0} = (\delta_\lambda X)u(\delta_\lambda g). \end{aligned}$$

In exponential coordinates $p = \exp(p_1 X_1 + \cdots + p_n X_n)$, we identify p with the n-uple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

In the sequel let $m_i = \dim V_i$ with $m_1 = m$. For a proof of the following Proposition see [50], Chapter 12.

Proposition 3.1.1. *In exponential coordinates the group product has the form*

$$x \cdot y = x + y + \mathcal{Q}(x, y) \qquad \forall x, y \in \mathbb{R}^n$$

where $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$ defined in (3.7), i.e.

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_i} \mathcal{Q}_i(x, y) \qquad \forall x, y \in \mathbb{G}.$$

Moreover $\forall x, y \in \mathbb{G}$

$$\begin{aligned} \mathcal{Q}_1(x, y) &= \ldots = \quad Q_m(x, y) = 0, \\ Q_j(x, 0) = Q_j(0, y) = 0 \qquad &and \qquad Q_j(x, x) = Q_j(x, -x) = 0, \quad m < j \leq n; \end{aligned}$$

finally, if $m_{i-1} < j \leq m_i$ and $2 \leq i$, then

$$Q_j(x, y) = \mathcal{Q}_j(x_1, \ldots, x_{m_{i-1}}, y_1 \ldots, y_{m_{i-1}}).$$

Exponential coordinates characterize the left invariant vector fields X_j as vector fields on $\mathbb{R}^n$.

Proposition 3.1.2. *The vector fields X_j have polynomial coefficents and moreover if $m_{i-1} < j \leq m_i$, $1 \leq l \leq s$,*

$$X_j(x) = \partial_j + \sum_{i>h^l}^{n} q_{i,j}(x)\partial_i,$$

where $q_{i,j}(x) = \dfrac{\partial \mathcal{Q}_i}{\partial y_j}(x,y)|_{y=0}$, so that if $m_{l-1} < i \leq m_l$ then $q_{i,j}(x) = q_{i,j}(x_1, \ldots, x_{m_{l-1}})$ and $q_{i,j}(0) = 0$.

Carnot groups are nilpotent and so unimodular, thus the right and the left Haar measures coincide, up to a constant multiples. We fix one of them and denote it by $\mathrm{vol}_{\mathbb{G}}$.

We shall denote by $\mathcal{H}^k$ the Hausdorff k-dimensional measure associated to the Carnot-Caratheodory distance on $\mathbb{G}$.

Returning to the example of the Heisenberg group, from (2.6) follows that, in canonical coordinates

$$d_c((0,0,0),(0,0,z)) \approx \sqrt{z},$$

thus d_c is, in general, not smooth so it is interesting to ask what is the Hausdorff dimension of $\mathbb{H}^1$ with respect to d_c. The answer follows by the so called ball-box theorem, for a proof see [42], [43].

Theorem 3.1.3 (Ball-box theorem)**.** *There exist continuous strictly positive functions $C_1(g)$, $C_2(g)$, on $\mathbb{G}$, such that*

$$C_1(g) \leq (vol_{\mathbb{G}} B_g(\rho))/\rho^Q \leq C_2(g) \qquad \forall \rho.$$

Consequently, the Hausdorff dimension of $\mathbb{G}$ with respect to the CC metric associated to $X_1, \ldots, X_m$ equals Q, where Q is the homogeneous dimension $\sum_i i\, dim V_i$.

The Hausdorff measure $\mathcal{H}^Q$ by the left translation and scaling invariance of the CC distance coincides with Haar measure on $\mathbb{G}$. Recall that Q is the homogeneous dimension of $\mathbb{G}$. Moreover, in exponential coordinates, all these measures coincide with a constant multiple of the Lebesgue measure on $\mathbb{R}^n$.

Proposition 3.1.4. *In exponential coordinates the Haar measure on* $\mathbb{G}$ *coincides with the Lebesgue* $\mathcal{L}^n$ *measure, i.e.*

$$vol_{\mathbb{G}}\left(\left\{exp(\sum_1^n x_iX_i) : (x_1,\dots,x_n)\in A\right\}\right) = c\mathcal{L}^n(A) \qquad \forall\, A\, \in\, \mathcal{B}(\mathbb{R}^n).$$

Proof. The proof is a consequence of the BCH formula (2.1.2). Clearly, for the uniqueness of the Haar measure, it is eqivalent to prove that $\mathcal{L}^n \circ exp^{-1}$ is invariant under right translations. Consider a set $B \subset \mathbb{R}^n$ and $h \in \mathbb{R}^n$. Then, by the BCH formula we have

$$exp^{-1}(exp(b)exp(h)) = c(b,h) \qquad b \in B$$

where $c(b,x) = b+h+c_1[b,h]+c_2[b,[b,h]]+\cdots+c_s[b,\dots[b,h]\dots]$, and the last commutator is iterated s-times (s is the step of the group $\mathbb{G}$). Note that, if we identify both $\mathfrak{g}$, $\mathbb{G}$ with $\mathbb{R}^n$, $c(b,x)$ is the product law of Proposition 3.1.1.
Denote by $c(B,h) = \{c(b,h)|\ b\in B\} \subset \mathbb{R}^n$, thus

$$\mathcal{L}^n(exp^{-1}(exp(B)exp(h))) = \mathcal{L}^n(c(B,h)).$$

Consider the linear map $c_h : b \mapsto c(b,h)$, if we prove that this map has Jacobian equal to 1 the proof is complete. Indeed, by Proposition 3.1.1 the matrix $\dfrac{\partial c(b,h)_j}{\partial b_i}$ is an upper triangular matrix with 1 on the diagonal, thus the determinant is 1.

□

From the previous Proposition and the definition of δ_λ easily follows:

$$\mathrm{vol}_{\mathbb{G}}(\delta_\lambda(A)) = \lambda^Q \mathrm{vol}_{\mathbb{G}}(A),$$

for all Borel sets $A \subseteq \mathbb{G}$.

The following result, from general geometric measure theory, will be useful in the sequel : for μ nonnegative Radon measure, $t>0$ and $B\subseteq\mathbb{G}$ Borel, we have

$$\limsup_{r\downarrow 0}\frac{\mu(B_r(x))}{\omega_k r^k} \geq t\ \forall x\in B \quad \Rightarrow \quad \mu(B)\geq t\mathcal{S}^k(B), \tag{3.11}$$

where ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$ and $\mathcal{S}^k$ (1.25) is the spherical k-dimensional Hausdorff measure. As a consequence we obtain that

$$\left\{x\in\mathbb{G} : \limsup_{r\downarrow 0}\frac{\mu(B_r(x))}{r^k} > 0\right\} \text{ is } \sigma\text{-finite with respect to } \mathcal{S}^k. \tag{3.12}$$

We know that Carnot groups are Ahlfors spaces and in order to apply the theorems in section (1.2) we need to prove a 1-Poincaré inequality on $\mathbb{G}$.

Proposition 3.1.5 ([30]). *Any Carnot group equipped with the Lebesgue measure and the Carnot-Carathéodory metric supports a* 1*-Poincaré inequality.*

Proof. Let $\mathbb{G}$ be a Carnot group with the CC metric denoted by d. Let u be a continuous function and g its upper gradient. It suffices to prove that

$$\int_B |u(x) - u_B|\, dx \leq Cr \int_{2B} g(x)\, dx,$$

on every ball of radius r. We can assume that the ball B is centered at 0. Set $|z| = d(0, z)$ and let $\gamma_z : [0, |z|] \to \mathbb{G}$ be a geodesic path that joins 0 with z, thus $s \mapsto x\gamma_z(s)$ is the shortest path that joins x to xz. Hence

$$|u(x) - u(xz)| \leq \int_0^{|z|} g(x\gamma_z(s))\, ds.$$

The left invariance of the Lebesgue measure yields

$$\begin{aligned} \int_B |u(x) - u_B|\, dx &\leq \frac{1}{|B|} \int_B \int_B |u(x) - u(y)|\, dydx \\ &= \frac{1}{|B|} \int_{\mathbb{G}} \int_{\mathbb{G}} \chi_B(x)\chi_B(xz)|u(x) - u(xz)|\, dxdxz \\ &\leq \frac{1}{|B|} \int_{\mathbb{G}} \int_{\mathbb{G}} \int_0^{|z|} \chi_B(x)\chi_B(xz)\, g(x\gamma_z(s))\, dsdxdz. \end{aligned}$$

From the right invariance of the measure we get

$$\begin{aligned} \int_{\mathbb{G}} \chi_B(x)\chi_B(xz)g(x\gamma_z(s))\, dx &= \int_{\mathbb{G}} \chi_{B\gamma_z(s)}(\xi)\chi_{Bz^{-1}\gamma_z(s)}(\xi)g(\xi)\, d\xi \\ &\leq \chi_{2B}(z) \int_{2B} g(\xi)\, d\xi, \end{aligned}$$

where we denoted by Bh the right translation of B by h. In the last inequality we used the following fact: if the second expression has nonzero value, then $\xi = x\gamma_z(s) = yz^{-1}\gamma_z(s)$ for some $x, y \in B$, thus $z = x^{-1}y \in 2B$. Consider the geodesic $x\gamma_{x^{-1}y}(t)$ which joins x with y, ξ lies on this geodesic and so $d(x, \xi) + d(y, \xi) = d(x, y)$, it follows that $d(y, \xi)$ or $d(x, \xi)$ is less or equal to r, which together with the triangle inequality implies $\xi \in 2B$.

From the previous inequality we get

$$\begin{aligned}\int_B |u(x)-u_B|\,dx &\le \frac{1}{|B|}\int_{\mathbb{G}}\int_0^{|z|}\chi_{2B}(z)\int_{2B} g(\xi)\,d\xi ds dz\\ &= \frac{1}{|B|}\int_{2B}\int_{2B}|z|g(\xi)\,d\xi dz\\ &\le Cr\int_{2B} g(\xi)\,d\xi,\end{aligned}$$

and the proof is complete. □

Remark 15. As a consequence of this we know that any Carnot group is doubling metric measure space which support a 1-Poincaré inequality, thus we can apply on it the theory of the first chapter.

3.2 Tangent Hyperplane in Carnot groups

In order to describe the structure of sets of finite perimeter in Carnot groups, the analysis of tangent spaces, in the spirit of De Giorgi's theorem, is a fundamental step.
Recent progress in this direction can be found in [24] and [4]. In step 2 groups there is a satisfactory theory of sets of locally finite perimeter. For general Carnot groups there isn't yet a complete description of the tangent space, recently L. Ambriosio, B. Kleiner and E. Le Donne in [4] proved that, given $E \subseteq \mathbb{G}$ of finte perimeter, at $|D\chi_E|$-a.e. points $x \in \partial^* E$, $Tan(E,x)$ contains at least an halfspace.
Elements of the tangent space $Tan(E,x)$ are blow-up sets of E and consequently they have constant outer normal (see [24]), so a classification of sets with constant normal will characterize the structure of $Tan(E,x)$. Unfortunately, in contrast with Euclidean spaces, there are sets of constant outer normal that aren't halfspaces. An example of this will be given at the end of the section, it will be a cone in the Engel group $\mathbb{E}$.
Here we state the main result of [4], for a complete proof we refer to the next section where a modification of it will be used to prove the locality property of Carnot groups.

3.2.1 X-derivative and sets of finite perimeter

Throughout this section, as in [4], we will denote by M a smooth differentiable manifold with topological dimension n, endowed with a n, differential volume form vol_M (later M

will be a Carnot Group $\mathbb{G}$, and $vol_{\mathbb{G}}$ the right Haar measure).
Given a vector field $X \in \Gamma(TM)$ we define the divergence as follows:

$$\int_M Xu\, d\,vol_M = -\int_M u\, \mathrm{div} X\, d\,vol_M \qquad \forall u \in C_c^\infty(M). \tag{3.13}$$

Definition 3.3. Let $u \in L^1_{loc}(M)$ and let $X \in \Gamma(TM)$ be divergence-free. We denote by Xu the distribution

$$\langle Xu, v\rangle := -\int_M uXv\, d\,vol_M, \qquad v \in C_c^\infty(M).$$

If $f \in L^1_{loc}(M)$, we write $Xu = f$ if $\langle Xu, v\rangle = \int_M vf\, d\,vol_M$ for all $v \in C_c^\infty(M)$. Analogously, if μ is a Radon measure in M, we write $Xu = \mu$ if $\langle Xu, v\rangle = \int_M v\, d\mu$ for all $v \in C_c^\infty(M)$.

Of course, if $u \in C^1(M)$ the distributional derivative coincides with the classical one, moreover if we work in an Euclidean space, the X-derivative of characteristic functions of regular domains can be computed.

Remark 16. Let $u \in C^1(\mathbb{R}^n)$, then $Xu = \langle X, \nabla u\rangle$. Assume now that $E \subset \mathbb{R}^n$ is locally the sub-level set of f, and let $X \in \Gamma(T\mathbb{R}^n)$ be divergence-free. Then, for any $v \in C_c^\infty(\mathbb{R}^n)$, using the Gauss-Green formula, we have

$$\int_E Xv\, dx = \int_{\partial E} \langle vX, \nu_E^{eu}\rangle\; d\,\mathcal{H}^{n-1}$$

where ν_E^{eu} is the unit (Euclidean) outer normal to E, this prove that

$$X\chi_E = -\langle X, \nu_E^{eu}\rangle\, \mathcal{H}^{n-1} \llcorner \partial E.$$

We can compute explicitly the formula for the outer normal to E, it is $\nu_E^{eu} = \nabla f(x)/|\nabla f(x)|$, so we get

$$\langle X, \nu_E^{eu}\rangle = \left\langle X, \frac{\nabla f}{|\nabla f|}\right\rangle = \frac{Xf}{|\nabla f|}.$$

Thus

$$X\chi_E = -\frac{Xf}{|\nabla f|}\mathcal{H}^{n-1} \llcorner \partial E. \tag{3.14}$$

Given $X \in \Gamma(TM)$ we denote by $\varphi_X : M \times \mathbb{R} \to M$ the flow of X.

Theorem 3.2.1 ([4]). *Let $u \in L^1_{loc}(M)$ be satisfying $Xu = 0$ in the sense of distributions. Then, for all $t \in \mathbb{R}$, $u = u \circ \Phi_X(\cdot, t)$ vol_M-a.e. in M.*

Remark 17. The flow is vol_M-measure preserving if and only if $\mathrm{div}X$ is equal to 0. Indeed, if $f \in C^1_c(M)$, the measure preserving property gives that $\int_M f(\Phi_X(x,t))\, d\,\mathrm{vol}_M$ is independent of t:

$$\begin{aligned} 0 = \int_M \frac{d}{dt} f(\Phi_X(x,t))\, d\,\mathrm{vol}_M(x) &= \int_M Xf(\Phi_X(x,t))\, d\,\mathrm{vol}_M(x) \\ &= \int_M Xf(y)\, d\,\mathrm{vol}_M(x). \end{aligned}$$

Therefore $\int_M f \mathrm{div}X\, d\,\mathrm{vol}_M(x) = 0$ for all $f \in C^1_c(M)$, and X is divergence-free. If X is divergence-free then for all $u \in C^\infty_c(M)$ we have $\int_M Xu\, d\,vol_M = 0$, clearly $u \circ \Phi_X(\cdot, t)$ is $C^1_c(M)$ then we get

$$0 = \int_M Xu \circ \Phi_X(\cdot, t)\, d\,\mathrm{vol}_M = \int_M \frac{d}{dt} u \circ \Phi_X(\cdot, t)\, d\,\mathrm{vol}_M. \qquad \forall u \in C^\infty_c(M)$$

Then $\int_M u \circ \Phi_X(\cdot, t)\, d\,\mathrm{vol}_M$ is constant in t, this implies

$$\int_M u\, d\,\mathrm{vol}_M(\Phi_X^{-1}(\cdot, t)) = \int_M u\, d\,vol_M \qquad \forall u \in C^\infty_c(M),$$

and the two measures coincide as distributions, but they are both measures thus the proof is complete.

Let $\mathbb{G}$ a Lie group, we denote by e the identity of the group, by $R_g(h) := hg$ the right translation, and by $L_g(h) := gh$ the left translation. We shall also denote $\mathrm{vol}_{\mathbb{G}}$ the volume form and the right-invariant Haar measure. We shall focus on the left invariant vector fields.

Remark 18. Let $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra associated to $\mathbb{G}$, and denote by $\exp(tX)$ the flow of X at time t starting from e. Let $\gamma(t) := \exp(tX)$ and $\gamma_g(t) := g\gamma(t)$, then

$$\frac{d}{dt}\gamma_g(t) = \frac{d}{dt}(L_g(\gamma(t))) = (dL_g)_{\gamma(t)} \frac{d}{dt}\gamma(t) = (dL_g)_{\gamma(t)} X = X_{\gamma_g(t)}.$$

This implies that $\Phi_x(\cdot, t) = R_{\exp(tX)}$ and so the flow preserves the right Haar measure, it follows that all $X \in \mathfrak{g}$ are divergence-free.

Now we recall the definition of the adjoint map. For $k \in \mathbb{G}$ the conjugation map C_k is the composition of L_k with $R_{k^{-1}}$. The *adjoint* representation of $\mathbb{G}$, Ad maps $\mathbb{G}$ in $Aut(\mathfrak{g})$ as follows

$$Ad_k(X) := (C_k)_* X \qquad Ad_k(X)f(x) = X(f \circ C_k)(C_k^{-1}(x)).$$

Proposition 3.2.2 ([4]). *Assume that $\mathbb{G}$ is a connected, simply connected nilpotent Lie group. Let $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$ satisfying $dim(\mathfrak{g}') + 2 \leq dim(\mathfrak{g})$, and assume that $W := \mathfrak{g}' \oplus \{\mathbb{R}X\}$ generate the whole Lie algebra of $\mathfrak{g}$ for some $X \notin \mathfrak{g}'$. Then, there exists $k \in exp(\mathfrak{g}')$ such that $Ad_k(X) \notin W$.*

Proof. Note that $\mathfrak{g}'$ is a finite-dimensional sub-algebra and that exp is, under the simple connectedness assumption, a homeomorphism, hence $\mathbb{K} := \exp(\mathfrak{g}')$ is a closed Lie subgroup of $\mathbb{G}$. Therefore, we can consider the quotient manifold $\mathbb{G}/\mathbb{K}$, in fact the homomogeneus space of right cosets: it consist of the equivalence classes of $\mathbb{G}$ induced by the relation

$$x \sim y \;\Leftrightarrow\; y^{-1}x \in \mathbb{K}.$$

Denote by $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$ the canonical projection. The natural topology of $\mathbb{G}/\mathbb{K}$ is determined by the requirement that π should be continuous and open. Let $\mathfrak{m}$ denote some vector space of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$. The sub-manifold $\exp(\mathfrak{m})$ is referred to as a local cross section for $\mathbb{K}$ at the origin, and it can be used to give a differentiable structure to $\mathbb{G}/\mathbb{K}$. In fact, let $Z_1, \ldots, Z_r$ be a basis of $\mathfrak{m}$, then the mapping

$$(x_1, \ldots, x_r) \mapsto \pi(g\exp(x_1 Z_1 + \cdots x_r Z_r)$$

is a homeomorphism of an open set of $\mathbb{R}^r$ onto a neighborhood of $g\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$. Then one can prove ([31]) that with these charts, $\mathbb{G}/\mathbb{K}$ is an analytic manifold. In particular, π restricted to $\exp(\mathfrak{m})$ is a local diffeomorphism into $\mathbb{G}/\mathbb{K}$ and $d\pi(X) \neq 0$ since the projection of X on $\mathfrak{m}$ is non zero. Notice that, by our assumption on the dimension of $\mathfrak{g}'$, the topological dimension of $\mathbb{G}/\mathbb{K}$ is at least 2. Consider the restriction of Ad to $\mathbb{K}$, it maps $\mathbb{K}$ in $Aut(\mathfrak{g}')$ and this implies that $Ad_k(\mathfrak{g}') \subseteq \mathfrak{g}'$. Now, if the statement were false we would have $Ad_k(W) \subseteq W$ for all $k \in \mathbb{K}$. By the definition of adjoint representation as composition of the differential of right and left translation, the above would be equivalent to

$$(R_k)_*((L_{k^{-1}})_*(Y)) \in W \qquad \forall Y \in W,\; k \in \mathbb{K}.$$

Since the vector fields in W are left invariant, this condition would say that W is $\mathbb{K}$-right invariant, and we can write this condition in the form $d(R_k)_x(W_x) \subset W_{xk}$ for all $x \in \mathbb{G}$ and $k \in \mathbb{K}$. Now, let us consider the subspaces $d\pi_x(W_x)$ of $T_{\pi(x)}\mathbb{G}/\mathbb{K}$: they are all 1-dimensional, thanks to the fact that $\dim(W) = 1+ \dim(\mathfrak{g}')$, and they depend only on $\pi(x)$: indeed, $\mathbb{K}$-right invariance and the identity $\pi \circ R_k = \pi$ gives

$$d\pi_x(Y_x) = d\pi_{xk}(d(R_k)_x(Y_x)) \in d\pi_{xk}(W_{xk})$$

for all $Y \in W$ and $k \in \mathbb{K}$. Therefore we can define a (smooth) 1-dimensional distribution $W/\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$ by $(W/\mathbb{K})_y := d\pi_x(W_x)$, where x is any element of $\pi^{-1}(y)$. In particular $W/\mathbb{K}$ would be tangent to a 1-dimensional foliation $\mathcal{F}$ of $\mathbb{G}/\mathbb{K}$ that has at least codimension 1, since $\mathbb{G}/\mathbb{K}$ has at at least dimension 2. Letting $\mathcal{F}'$ be the foliation of $\mathbb{G}$ whose leaves are the inverse images via π of leaves of $\mathcal{F}$, we find that still $\mathcal{F}'$ has codimension at least 1, and W is tangent to the leaves of $\mathcal{F}'$. But this contradicts the fact that W generates $\mathfrak{g}$: in fact, the only sub-manifold to which W could be tangent is all the manifold $\mathbb{G}$. $\square$

The next proposition is an explicit characterization of the set spanned by $Ad_{\exp(Y)}(X)$ where Y varies in a subalgebra of $\mathfrak{g}$, we prove it for completeness, but we won't use it in the sequel.

Proposition 3.2.3 ([4]). *Let $\mathfrak{g}$ be a nilpotent Lie algebra, let $\mathfrak{g}' \subset \mathfrak{g}$ be a Lie algebra and let $X \in \mathfrak{g}$. Then*

$$\mathit{span}\left(\left\{Ad_{exp(Y)}(X) : Y \in \mathfrak{g}'\right\}\right) = [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \cdots .$$

Proof. Let us denote by S the space $\mathrm{span}\left(\left\{Ad_{exp(Y)}(X) : \ Y \in \mathfrak{g}'\right\}\right)$, S contains X and all vector fields $Ad_{exp(Y)}$ for $r \geq 0$ and $Y \in \mathfrak{g}'$. Let us recall the forumula $Ad_{exp(Y)} = e^{ad_Y}$ where $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the operator $ad_Y(X) = [Y, X]$. Therefore

$$Ad_{exp(Y)}X = X + [Y, X] + \frac{1}{2}[Y, [Y, X]] + \cdots$$

Let ν be the dimension of $\mathfrak{g}'$ and let $(Y_1, \ldots, Y_\nu)$ be a basis of $\mathfrak{g}'$. For all $Y = \sum_1^\nu r_j Y_j \in \mathfrak{g}'$

and taking into account the previous equation, define

$$\begin{aligned}\Phi(r_1,\ldots,r_\nu) &:= Ad_{exp(\sum_1^\nu r_j Y_j)}X - X\\ &= \sum_{k=1}^{s-1}\frac{1}{k!}\left(\sum_{j=1}^{\nu} ad\,Y_j\right)^k X\\ &= \sum_{k=1}^{s-1}\frac{1}{k!}\sum_{j_1,\ldots,j_k=1}^{\nu} r_{j_1}\cdots r_{j_k}\,(ad\,Y_j 1\cdots ad\,Y_j k)\,X\in S.\end{aligned}$$

Since this polynomial takes values in S, it turns out that all its coefficients belong to S. In particular we have

$$ad\,Y_i(X)=\partial_{r_j}\Phi(0)\in S \qquad \text{and}$$
$$(ad\,Y_i ad\,Y_j + ad\,Y_j ad\,Y_i)X = 2\partial_{r_i}\partial_{r_j}\Phi(0)\in S.$$

The Jacobi identity can be read as $ad_U ad_V - ad_W ad_U = ad_{[U,W]}$ so that

$$(ad\,Y_i ad\,Y_j + ad\,Y_j ad\,Y_i)X = 2ad\,Y_i ad\,Y_j\,X + ad_{[Y_i,Y_j]}X.$$

It follows that $(ad\,Y_i ad\,Y_j)X\in S$, and this prove that $[\mathfrak{g}',X]+[\mathfrak{g}',[\mathfrak{g}',X]]\subset S$. By induction, let us suppose that

$$\mathfrak{u}_{k-1} := [\mathfrak{g}',X]+[\mathfrak{g}',[\mathfrak{g}',X]]+\cdots+[\mathfrak{g}',[\mathfrak{g}',\ldots,[\mathfrak{g}',X]]]\subset S$$

where the last element of the sum is a $(k-1)$-order commutator, for some $k\geq 3$. In general we have

$$\partial_{r_{i_1}}\cdots\partial_{r_k}\Phi(0)=\frac{1}{k!}\left(\sum_{\sigma}(ad\,Y_{j_{\sigma(1)}}\cdots ad\,Y_{j_{\sigma(k)}}\right)S\in S, \tag{3.15}$$

where the sum runs on all permutations σ of k elements. By the Jacobi identity

$$\left(ad\,Y_{j_{\sigma(1)}}\cdots ad\,Y_{j_{\sigma(k)}}\right)X-\left(ad\,Y_{j_{\eta(1)}}\cdots ad\,Y_{j_{\eta(k)}}\right)X\in\mathfrak{u}_{k-1}$$

if $\sigma\circ\eta^{-1}$ is a transposition. Then, by the inductive assumption, we can iterate transpositions in $\left(ad\,Y_{j_{\sigma(1)}}\cdots ad\,Y_{j_{\sigma(k)}}\right)X$ to write it as

$$(ad\,Y_{j_1}\cdot sad\,Y_{j_k})\,X+W_\sigma \quad \text{with} \quad W_\sigma\in S.$$

Then, from (3.15) we get $(ad\,Y_{j_1}\cdots ad\,Y_{j_k})\,X\in S$, so that $\mathfrak{u}_k\subset S$, and the theorem follows by induction. $\square$

Definition 3.4. Let $f \in L^1_{loc}(\mathbb{G})$. We shall denote by $Reg(f)$ the vector subspace of $\mathfrak{g}$ made by vectors X such that Xf is representable by a Radon measure. We shall denote by $Inv(f)$ the subspace of $Reg(f)$ corresponding to the vector fields X such that $Xf = 0$, and by $Inv_0(f)$ th subset made by homogeneous directions, i.e.

$$Inv_0(f) := Inv(f) \cap \bigcup_{i=1}^{s} V_i.$$

We will consider regular and invariant directions of characterisctic functions, therefore we set

$$Reg(E) := Reg(\chi_E), \quad Inv(E) := Inv(\chi_E) \quad Inv_0(E) := Inv_0(\chi_E).$$

We now define halfspaces in $\mathbb{G}$, as subsets of $\mathbb{G}$ having invariance along a codimension 1 space of directions, and monotonicity along the remaining one.

Definition 3.5. We say that a Borel set $H \subseteq \mathbb{G}$ is a vertical halfspace if $Inv_0(H) \supseteq \cup_2^s V_i$, $V_1 \cap Inv_0(H)$ is a codimension one subspace of V_1 and $X\chi_H \geq 0$ for some $X \in V_1$.

Indeed, identifying the Lie algebra $\mathfrak{g}$ with $\mathbb{R}^n$, vertical halfspaces are images by the exponential maps of halfspaces in $\mathbb{R}^n$, as stated in the following proposition.
Recall that m denotes the dimension of V_1 and $X_1, \ldots, X_m$ is a given orthonormal basis of V_1.

Proposition 3.2.4. *$H \in \mathbb{G}$ is a vertical halfspace if and only if there exist $c \in \mathbb{R}$ and a unit vector $\nu \in \mathbf{S}^{m-1}$ such that $H = H_{c,\nu}$, where*

$$H_{c,\nu} := exp\left(\left\{\sum_{i=1}^{m} a_i X_i + \sum_{i=2}^{s} v_i : \ v_i \in V_i, \ a \in \mathbb{R}^n, \ \sum_{i=1}^{m} a_i \nu_i \leq c\right\}\right). \tag{3.16}$$

Proof. Denote by $\nu \in S^{m-1}$ the unique vector such that the vector $Y = \sum_i \nu_i X_i$ is orthogonal to all invariant directions in V_1. Let us work in graded coordinates, with the function

$$(x_1, \ldots, x_n) \mapsto exp(\sum_{i=1}^{n} x_i v_i)$$

and let $\widetilde{H} \subset \mathbb{R}^n$ be the set H in these coordinates. Here $(v_1, \ldots, v_n)$ is a basis of $\mathfrak{g}$ compatible with the stratification: if m_i is the dimensions of V_i, $1 \leq i \leq s$, $l_0 = 0$ and

$l_i = \sum_1^i m_j$, then $v_{l_{i-1}+1}, \dots, v_{l_i}$ is a basis of V_i. In these coordinates the vector fields v_i, by the BCH formula (2.1.2), correspond to ∂_{x_i} for $l_{s-1}+1 \le i \le l_s = n$, and Theorem 3.2.1 gives that $\chi_{\widetilde{H}}$ does not depend on $x_{l_{s-1}+1}, \dots, x_n$. For $l_{s-2}+1 \le i \le l_{s-1}$ the vector fields $v_i - \partial_{x_i}$, are given by the sum of polynomials multiplied by ∂_{x_j}, with $l_{s-1}+1 \le j \le l_s$. Fix $l_{s-2}+1 \le i \le l_{s-1}$and consider $v_i - \partial_{x_i} = \sum_j p_j(x)\partial_{x_j}$, with $l_{s-1}+1 \le j \le l_s$. Let ϕ_ϵ, $\epsilon > 0$ the classical euclidean mollifiers and define $\chi_\epsilon := \chi_{\widetilde{H}} * \phi_\epsilon$. Now we compute $p_j(x)\partial_{x_j}\chi_\epsilon$

$$p_j(x)\partial_{x_j}\chi_\epsilon = p_j(x)\int_{\mathbb{R}^n} \chi_{\widetilde{H}}(y)\epsilon^{-n}\partial_{x_j}\phi(\frac{x-y}{\epsilon})\,dy$$

the right hand side of the previous equation is zero, in fact $\partial_{x_j}\chi_{\widetilde{H}} = 0$ in the distributional sense. The above consideration and $v_j \in Inv(\widetilde{H})$ implies $\partial_{x_i}\chi_\epsilon = 0$, then passing to the limit in $\epsilon \downarrow 0$ we get $\partial_{x_i}\chi_{\widetilde{H}} = 0$ in the sense of distributions and we can apply Theorem 3.2.1 to obtain that $\chi_{\widetilde{H}}$ does not depend on $x_{l_{s-2}+1}, \dots, x_{l_{s-1}}$ either. If we continue in this way we obtain that $\chi_{\widetilde{H}}$ depends on $(x_1, \dots, x_m)$ only. Moreover, $\sum_i \xi_i\partial_{x_i}\chi_{\widetilde{H}}$ is equal to 0 if $\xi \perp \nu$, and it is nonnegative if $\xi = \nu$. Then applying the same argument that appears in De Giorgi's rectifiability proof [17] we have that $\chi_{\widetilde{H}}$ depends on $\sum_1^m \nu_i x_i$ only, and it is a monotone function of this quantity, so (3.16) is proved. □

We define now the class of sets of locally finite perimeter :

Definition 3.6. A Borel set $E \subset G$ has locally finite perimeter if $X\chi_E$ is a Radon measure for any $X \in V_1$. Given a set E of locally finite perimeter, we denote by $D\chi_E$ the vector valued measure

$$D\chi_E = (X_1\chi_E, \dots, X_m\chi_E).$$

Definition 3.7 (De Giorgi's reduced boundary)**.** Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. We denote by $\mathcal{F}(E)$ the set of points $x \in \operatorname{supp}|D\chi_E|$ where:

1. the limit $\nu_E(x) = (\nu_{E,1}(x), \dots, \nu_{E,m}(x)) := \lim_{r\downarrow 0} \dfrac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$ exists,
2. $|\nu_E(x)| = 1$.

We know that Carnot groups are Ahlfors spaces (see Definition 1.8), moreover, by Proposition (3.1.5), they support a 1-Poincaré inequality, thus applying Theorems 1.2.2 and 1.2.3 and Corollary 1.2.4 to the case of Carnot groups we obtain the following important theorem :

Theorem 3.2.5. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then $|D\chi_E|$ is asymptotically doubling, and more precisely the following property holds: for $|D\chi_E|$-a.e. $x \in \mathbb{G}$ there exist $\overline{r}(x) > 0$ satisfying*

$$l_{\mathbb{G}} r^{Q-1} \leq |D\chi_E|(B_r(x)) \leq L_{\mathbb{G}} r^{Q-1} \qquad \forall r \in (0, \overline{r}(x)), \tag{3.17}$$

with $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ depending on $\mathbb{G}$ only. As a consequence $|D\chi_E|$ is concentrated on $\mathcal{F}E$, i.e., $|D\chi_E|(\mathbb{G} \setminus \mathcal{F}E) = 0$.

3.2.2 The Tangent set

Definition 3.8. Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \mathcal{F}E$. Denote by $\mathrm{Tan}(E, x)$ all limit points, in the topology of local convergence in measure, of the translated and rescaled family of sets $\left\{\delta_{\frac{1}{r}}(x^{-1}E)\right\}_{r>0}$ as $r \downarrow 0$.
If $F \in \mathrm{Tan}(E, x)$ we say that F is tangent to E at x. We also set

$$\mathrm{Tan}(E) := \bigcup_{x \in \mathcal{F}E} \mathrm{Tan}(E, x).$$

The main result of this section is the following theorem, for a proof see ([4]) or the next section.

Theorem 3.2.6. *Suppose $E \subseteq \mathbb{G}$ has locally finite perimeter. Then, for $|D\chi_E|$-a.e. $\overline{x} \in \mathbb{G}$ a vertical halfspace H belongs to the tangent set to E at $\overline{x}$.*

Now we give an example of a set of locally finite perimeter with constant outer normal which is different from an halfspace.

Example 9 (The Engel Group [4]). Let $\mathbb{E}$ be the Carnot group whose Lie algebra is $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = span\{X_1, X_2\}$, $V_2 = \{\mathbb{R}X_3\}$ and $V_3 = \{\mathbb{R}X_4\}$, with the relations

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = -X_4.$$

A possible representation of the vector fields in $\mathbb{R}^n$ is

$$\begin{aligned} X_1 &= \partial_1, \\ X_2 &= \partial_2 - x_1\partial_3 + \tfrac{x_1^2}{2}\partial_4, \\ X_3 &= \partial_3 - x_1\partial_4, \\ X_4 &= \partial_4. \end{aligned}$$

$\mathbb{E}$ is a Carnot group of step 3 with horizontal layer of dimension $m = 2$, it has topological dimension 4 and homogeneous dimension $Q = 7$.
Fix an $\alpha > 0$, let $P = P_\alpha : \mathbb{R}^4 \to \mathbb{R}$ be the polynomial

$$P(x) = \alpha x_2^3 + 2x_4, \qquad \nabla P(x) = (0, 3\alpha x_2^3, 0, 2).$$

We can compute easly the derivatives of P along the vector fields $X_i,\ i = 1, 2$:

$$X_1 P(x) = 0, \qquad X_2 P(x) = 3\alpha x_2^2 + x_1^2 \geq 0. \tag{3.18}$$

Consider the set $C := \{x \in \mathbb{R}^4 :\ P(x) \leq 0\}$, whose boundary ∂C is the level set $\{P = 0\}$, due to the homogeneity of the Engel C is a cone, i.e. $\delta_r C = C$ for all $r > 0$.
The Euclidean outer normal of C is $\nu_C^{eu}(x) = \nabla P(x)/|\nabla P(x)|$, clearly C is a set of locally finite perimeter and by (3.14) we have

$$Z\chi_C = -\frac{ZP}{|\nabla P|}\mathcal{H}^3 \llcorner \partial C \qquad \forall Z \in \mathfrak{g}.$$

In particular by (3.18) we get

$$D\chi_C = (X_1\chi_C, X_2\chi_C) = (0,1)X_2\chi_C = -\frac{x_1^2 + 3\alpha x_2^2}{|\nabla P(x)}(0,1)\mathcal{H}^3 \llcorner \partial_C.$$

thus

$$|D\chi_C| = \frac{x_1^2 + 3\alpha x_2^2}{|\nabla P(x)}(0,1)\mathcal{H}^3 \llcorner \partial_C.$$

3.3 Locality property of Carnot groups

Here we prove that Carnot groups are local spaces in the sense of Definition (1.11). The proof will follows using the techniques developed in [4].
If X is a $\mathcal{U}$-space, given $v \in BV_{loc}(X)$ one can prove a nice decomposition of the measure $|Dv|$ (see Theorem 1.3.10), thus we can apply this to the case of Carnot groups and show that if $v \in BV_{loc}(\mathbb{G})$ and $\psi \in \Lambda$ (see 1.38) then

$$|D(\psi \circ u)| = \psi'(\tilde{u})|D^d u| + \Psi(u) S^h \llcorner S_u$$

here $\Psi(u)$ is

$$\Psi(u) = [\psi(u^\vee) - \psi(u^\wedge)]\,\Theta_u$$

and Θ_u is a function defined on S_u.

Proposition 3.3.1 ([4])**.** *Let $f \in L^1_{loc}(\mathbb{G})$. Then $Reg(f)$, $Inv(f)$, $Inv_0(f)$ are invariant under left translations, and $Inv_0(f)$ is invariant under intrinsic dilations. Moreover:*

1. *$Inv(f)$ is a Lie subalgebra of $\mathfrak{g}$ and $[Inv_0(f), Inv_0(f)] \subset Inv_0(f)$,*
2. *If $X \in Inv(f)$ and $k = exp(X)$, then Ad_k maps $Reg(f)$ into $Reg(f)$ and $Inv(f)$ into $Inv(f)$, i.e.*
$$Ad_k(Y)f = (R_{k_{-1}})_\sharp Y f \qquad \forall Y \in Reg(f).$$

Proof. For all $X, Y \in Inv(f)$ we have

$$\int_{\mathbb{G}} f[X,Y]g\, d\, vol_{\mathbb{G}} = -\langle Xf, Yg\rangle + \langle Yf, Xg\rangle = 0 \qquad \forall g \in C_c^\infty(\mathbb{G})$$

The second property follows from the graded structure of the algebra. Let $Y \in Reg(f)$ and $Z = Ad_k(Y)$. For $g \in C_c^\infty(\mathbb{G})$ and $k \in \mathbb{G}$, the left invariance of Y gives

$$\begin{aligned} Zg(x) &= Y(g \circ C_k)(C_k^{-1}(x)) = Y(g \circ R_{k_{-1}})(L_k \circ C_k^{-1}(x)) \\ &= Y(g \circ R_{k_{-1}})(R_k(x)). \end{aligned}$$

Therefore $(Zg) \circ R_{k_{-1}} = Y(g \circ R_{k_{-1}})$ and we have

$$\int_{\mathbb{G}} f Z g\, d\, vol_{\mathbb{G}} = \int_{\mathbb{G}} f \circ R_{k_{-1}} Y(g \circ R_{k_{-1}})\, d\, vol_{\mathbb{G}}.$$

If $k = exp(X)$ with $X \in Inv(f)$, we have $f \circ R_{k_{-1}} = f$ and the Proposition is proved. □

In this section $E, F \subseteq \mathbb{G}$ are sets of finite perimeter, following [4] we first prove that at $\mathcal{H}^{Q-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$, the outer normals coincide, i.e. $\nu_E(x) = \nu_F(x)$. The locality property (1.11) follows from a blow-up argument.
Now we generalize the definition (3.8) of tangent space, considering couples of blow-up at a common scale.

Definition 3.9. Let $E, F \subseteq \mathbb{G}$ sets of locally finite perimeter, given $x \in \partial^* E \cap \partial^* F$ we denote by $Tan(E, F, x)$ all limit points, in the topology of local convergence in measure, of the translated and rescaled family of sets pair $(\delta_{1/r}(x^{-1}E), \delta_{1/r}(x^{-1}F))$ as $r \downarrow 0$.
If $(T_1, T_2) \in Tan(E, F, x)$ we say that (T_1, T_2) is tangent to (E, F) at x. We also set

$$Tan(E, F) := \bigcup_{x \in \partial^* E \cap \partial^* F} Tan(E, F, x).$$

By Theorem 3.16, we need only to consider points $x \in \mathcal{F}E \cap \mathcal{F}F$.

The following Proposition is a slight modification of Theorem 3.1 in [24], where it is shown that tangent sets at points in the reduced boundary are invariant along a codimension 1 subspace of V_1 and monotone along the other direction.

Theorem 3.3.2. *Let $E, F \subseteq \mathbb{G}$ be sets of locally finite perimeter. Then, $\mathcal{H}^{Q-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ the following properties hold:*

1. $0 < \liminf_{r\downarrow 0} |D\chi_E|(B_r(x))/r^{Q-1} \leq \limsup_{r\downarrow 0} |D\chi_E|(B_r(x))/r^{Q-1} < \infty$;
2. *$Tan(E, F, x) \neq \emptyset$ and for all $(E_1, F_1) \in Tan(E, F, x)$ we have that $e \in supp|D\chi_{E_1}|$, $e \in supp|D\chi_{F_1}|$, and*

$$\nu_{E_1} = \nu_E(x) \qquad |D\chi_{E_1}| - a.e. \qquad \nu_{F_1} = \nu_F(x) \qquad |D\chi_{F_1}| - a.e.\ in\ \mathbb{G}.$$

In particular $V_1 \cap Inv_0(E_1)$ coincide with the codimension 1 subspace of V_1

$$\left\{ \sum_{i=1}^{m} a_i X_i : \sum_{i=1}^{m} a_i \nu_{E,i}(x) = 0 \right\}$$

and, setting $X_x := \sum_{i=1}^{m} \nu_{E,i}(x)(X_i)_x \in \mathfrak{g}$, $X\chi_{E_1}$ is a non negative Radon measure. The same holds changing E_1 with F_1 and E with F.

Proof. First we prove that $Tan(E, F, x)$ is not empty and that elements of the tangent space have constant normal. Denote by E_r and F_r the sets $E_r = \delta_{1/r}(x^{-1}E)$, $F_r = \delta_{1/r_j}(x^{-1}F)$. Fix an $R > 0$, and let $x = 0$ then the homogeneity of the perimeter gives

$$|D\chi_{E_r}|(B(0, R)) = \frac{|D\chi_E|(B(0, rR))}{r^{Q-1}} \quad \forall r > 0,$$

and the same for F. By Theorem 3.2.5 the right hand side is bounded by a constant independent on $r < C/R$. It follows that for $R > 0$ and $0 < r < C/R$

$$|D\chi_{E_r}|(B(0, R)) \leq C' \qquad |D\chi_{F_r}|(B(0, R)) \leq C'.$$

Moreover we have the bound on the L^1 norm of χ_{E_r} and χ_{F_r}, i.e.

$$\|\chi_{E_r}\|_{L^1(B(0,R))} \leq \mathcal{L}^n(B(0, R)) \qquad \|\chi_{F_r}\|_{L^1(B(0,R))} \leq \mathcal{L}^n(B(0, R))$$

Hence by the compactness theorem (see [41]), there exist a sequence $r_j \downarrow 0$ and a function in $BV_{loc}(\mathbb{G})$ such that

$$\chi_{E_{r_j}} \to f \quad in\ L^1_{loc}(\mathbb{G}).$$

We can use again the compactness theorem to the sequence r_j to obtain a subsequence $r_{j_k} = s_k$ and a function h such that

$$\chi_{F_{s_k}} \to h \quad in\ L^1_{loc}(\mathbb{G}).$$

Passing to a subsequence, we may assume that $\chi_{E_{s_k}}$ and $\chi_{F_{s_k}}$ converge pointwise $vol_{\mathbb{G}}$-a.e. to f and h respectively. Therefore $f = \chi_{E_1}$ and $h = \chi_{F_1}$ for suitable measurable sets $E_1, F_1 \subset \mathbb{G}$. By the semicontinuity of the perimeter we have that E_1 and F_1 are sets of locally finite perimeter. Then $(E_1, F_1) \in Tan(E, F, x)$.

Consider E_{s_k}, the same result holds for F_{s_k}, since $\chi_{E_{s_k}} \to \chi_{E_1}$ in L^1_{loc} we have

$$\lim_{k\to\infty} \int_{E_{s_k}} div_{\mathbb{G}}\phi \, dx = \int_{E_1} div_{\mathbb{G}}\phi \, dx,$$

for any $\phi \in C^1_0(\mathbb{G}, H\mathbb{G})$. Moreover we have

$$\lim_{k\to\infty} \int_{\mathbb{G}} \left\langle \phi, \nu_{E_{s_k}} \right\rangle \, d|D\chi_{E_{s_k}}| = \int_{\mathbb{G}} \left\langle \phi, \nu_{E_1} \right\rangle \, d|D\chi_{E_1}|$$

since E_{s_k} and E_1 are sets of locally finite perimeter. Note that we can extend the previous relation to all $\phi \in C^0_c$, this follows by the uniform bound on the perimeter measure of E_{s_k} and the density of C^1_0 in C^0_0 with respect to the $\|\cdot\|_\infty$. Then extracting a subsequence we have

$$\nu_{E_{s_k}} |D\chi_{E_{s_k}}| \rightharpoonup \nu_{E_1} |D\chi_{E_1}|, \tag{3.19}$$

weakly as vector valued Radon measure. Again by the homogeneity of the perimeter

$$\int_{B(0,R)} \left\langle \phi, \nu_{E_{s_k}} \right\rangle \, d|D\chi_{E_{s_k}}| = \frac{1}{s_k^{Q-1}} \int_{B(0,s_k R)} \left\langle \phi \circ \delta_{1/s_k}, \nu_E \right\rangle \, d|D\chi_E|,$$

choose ϕ with $supp\phi \supset B(0, R+\delta)$, $\phi_{|B(0,R+\delta)} = e_j$ then one has

$$\frac{1}{|D\chi_{E_{s_k}}|(B(0,R))} \int_{B(0,R)} \nu_{E_{s_k}} \, d|D\chi_{E_{s_k}}| = \frac{1}{|D\chi_E|(B(0,s_k R))} \int_{B(0,s_k R)} \nu_E \, d|D\chi_E|.$$

As $k \to \infty$ the right hand side has limit $\nu_E(0)$, hence

$$\frac{1}{|D\chi_{E_{s_k}}|(B(0,R))} \int_{B(0,R)} \left\langle \nu_E(0), \nu_{E_{s_k}} \right\rangle \, d|D\chi_{E_{s_k}}| = (1 + o(1)). \tag{3.20}$$

The uniform bound on the perimeter measures implies $|D\chi_{E_{s_k}}| \rightharpoonup \lambda$, here λ is a Radon measure such that $\lambda \geq |D\chi_{E_1}|$. The coarea formula implies $\lambda(\partial B(0,R)) = 0$ $\mathcal{L}^1$-a.e $R > 0$, thus we can apply Proposition 1.62 (2) in [5] to get

$$\nu_{E_{s_k}} |D\chi_{E_{s_k}}|(B(0,R)) \to \nu_{E_1} |D\chi_{E_1}|(B(0,R)).$$

By the semicontinuity of the perimeter, (3.20) and the previous relation one has

$$|D\chi_{E_1}|(B(0,R)) \leq \liminf_{k\to\infty} |D\chi_{E_{s_k}}|(B(0,R)) \leq \int_{B(0,R)} \langle \nu_E(0), \nu_{E_1} \rangle \; d|D\chi_{E_1}|.$$

Thus $| \left\langle \nu_{\nu_E(0)}, \nu_{E_1}(x) \right\rangle | = 1$, for $|D\chi_{E_1}|$-a.e $x \in \mathbb{G}$.
The proof of the codimension property and the monotonicity in the other directions can be found in [24]. $\square$

Remark 19. By the definition or $Tan(E, F, x)$ it follows directly that $(E_1, F_1) \in Tan(E, F, x)$ then $E_1 \in Tan(E, x)$ and $F_1 \in Tan(F, x)$, conversely if $E_1 \in Tan(E, x)$ and $F_1 \in Tan(F, x)$ are blow ups at common scale follows that $(E_1, F_1) \in Tan(E, F, x)$. Thus if we prove a property that holds for every element of $Tan(E, x)$ and $Tan(F, x)$ then we can extend it to $Tan(E, F, x)$. We will use this fact in order to simplify the following proofs.

Consider a function $f \in L^1_{loc}(\mathbb{G})$ and $X \in \mathfrak{g}$. Then, for all $r > 0$ we have the following identity

$$\delta_{1/r} X(f \circ \delta_r) = r^{-Q} (\delta_{1/r})_\sharp (Xf) \tag{3.21}$$

in the sense of distributions. Define $X_r := \delta_r X$, if $g \in C_c^\infty(\mathbb{G})$ from (3.10) we get $X_r(g \circ \delta_r) = (Xg) \circ \delta_r$ and thus

$$\begin{aligned}
\langle X_r(f \circ \delta_r), g \rangle &= -\int_{\mathbb{G}} (f \circ \delta_r) X_r g \, d\,vol_{\mathbb{G}} = -\int_{\mathbb{G}} f (X_r g) \circ \delta_{1/r} \, d\,vol_{\mathbb{G}} \\
&= -r^{-Q} \int_{\mathbb{G}} f X(g \circ \delta_{1/r}) \, d\,vol_{\mathbb{G}} = \left\langle r^{-Q} (\delta_{1/r})_\sharp (Xf), g \right\rangle .
\end{aligned}$$

Lemma 3.3.3. *Let $E, F \subseteq \mathbb{G}$ as above, let $X \in Reg(E)$ and $Y \in Reg(F)$ and set $\mu = X\chi_E$, $\lambda = Y\chi_F$. Suppose that $X = \sum_{i=1}^l v_i$ and $Y = \sum_{i=0}^h w_i$, where $v_i, w_i \in V_i$. Then, for $\mathcal{H}^{Q-1}$-a.e. $x \in \partial^* E \cap \partial^* F$, $v_l \in Inv_0(T_1)$ and $w_h \in Inv_0(T_2)$ for all $(T_1, T_2) \in Tan(E, F, x)$.*

Proof. From (3.12) the set of points N where $\sup_{r\downarrow 0} r^{2-Q}(|\mu|+|\lambda|)(B_r(x))$ is positive is σ-finite with respect to $\mathcal{S}^{Q-2}$, so it is $\mathcal{S}^{Q-1}$ negligible and then $|D\chi_E|$, $|D\chi_F|$ negligible. We prove that the statement holds for any $x \in (\mathcal{F}E \cap \mathcal{F}F) \setminus N$. We can assume, without loss of generality, that $x = e$. Let $g \in C^1_c(\mathbb{G})$, and $R > 0$ such that $supp(g) \subsetneq B_R(e)$. Fix $(E_1, F_1) \in Tan(E, F, e)$, by Remark 19 it is enough to prove it for any $E_1 \in Tan(E, x)$ and $F_1 \in Tan(F, x)$, we will consider only $Tan(E, x)$.
Setting $X_r = r^l \delta_{1/r} X$ we have, by (3.21)

$$\int_{\mathbb{G}} \chi_{\delta_{1/r}E} X_r g \, d\, vol_{\mathbb{G}} \;\; = \;\; r^{l-Q} \int_{\mathbb{G}} g \circ \delta_{1/r} \, d\mu$$

Since $l \geq 2$ the right hand part of the previous equation is bounded by

$$\sup |g| r^{l-Q} \mu(B_{rR}(e)) \;\; = \;\; o(1).$$

Now, notice that $X_r \to v_l$ as $r \downarrow 0$ and choose a sequence $r_j \downarrow 0$ such that $\delta_{1/r_j}E \to E_1$, then we obtain $v_l \chi_{E_1} = 0$, with the same method $w_h \chi_{F_1} = 0$ and the Lemma is proved. □

As in [4] we define the iterated tangent spaces.

Definition 3.10. Let $x \in \mathcal{F}E \cap \mathcal{F}F$ we define $Tan^1(E, F, x) := Tan(E, F, x)$ and

$$Tan^{k+1}(E, F, x) := \bigcup_x \left\{ Tan(E_1^k, F_1^k) : \; (E_1^k, F_1^k) \in Tan^k(E, F, x) \right\}$$

The following Lemma is a fundamental step in order to prove the locality property, with it we can start an iterating process that proves the existence of a couple of vertical halfspaces in the tangent set $Tan(E, F, x)$.

Lemma 3.3.4. *Let $F, E \subseteq \mathbb{G}$ be sets of locally finite perimeter such that*

$$dim(span(Inv_0(E))) \leq n-2, \qquad dim(span(Inv_0(F))) \leq n-2.$$

and assume that $Inv(E)$, $Inv(F)$ have codimension 1 in V_1. Then $\mathcal{H}^{Q-1}$-a.e $x \in \mathcal{F}E \cap \mathcal{F}F$ there exists $(E_2, F_2) \in Tan^2(E, F, x)$ such that

$$Inv_0(E_2) \supsetneq Inv_0(E), \qquad Inv_0(F_2) \supsetneq Inv_0(F).$$

Proof. Notice that by Remark 19 it is enough to consider only elements of $Tan(E, x)$. First we prove that there exists

$$Z \in \mathfrak{g} \setminus span(Inv_0(E)) + V_1,$$

such that $Z \in Reg(E_1)$, for all $E_1 \in Tan(E, x)$.
Applying Proposition 3.2.2 with $\mathfrak{g}' = span(Inv_0(E))$ and $X = \sum_1^m v_{E,i}(x)X_i$, we obtain $X' \in \mathfrak{g}'$, such that

$$Z := Ad_{exp(X')}(X) \notin span(Inv_0(E)) \oplus \{\mathbb{R}X\} = span(Inv_0(E)) + V_1,$$

where the equality follows by the codimension 1 property. Notice that $X' \in Inv(E)$, since $Inv_0(E) \subset Inv_0(E_1)$ we have that $X' \in Inv(E_1)$. Therefore Proposition 3.3.1 (2) shows that $Z \in Reg(E_1)$.

Take Z as above and recall that $Z \in Reg(E)$, choose a point $\bar{x} \in \partial^* E \cap \partial^* F$ such that Lemma 3.3.3 holds. Set $\mu = Z\chi_{E_1}$, removing from Z the horizontal component that belongs to $Reg(E)$ we can write

$$Z = v_{i_1} + \cdots + v_{i_l} \quad i_j \geq 2 \quad v_{i_j} \in V_{i_j}.$$

Then by Lemma 3.3.3 $v_{i_l} \in Inv_0(E_1)$ for all $E_1 \in Tan(E, x)$, it follows that $Z - v_{i_l} \in Reg(E_1)$, $E_1 \in Tan(E, x)$. Choose the biggest i_k such that $v_{i_k} \notin Inv_0(E_1)$ for some $E_1 \in Tan(E, x)$, consider know $Z' = v_{i_1} + \cdots + v_{i_k}$. $Z'\chi_{E_1}$ is still a measure $Z \in Reg(E_1)$ and $v_{i_{k+1}} + \cdots + v_{i_l} \in Inv_0(E_1)$ using the consideration above, clearly $Z'\chi_{E_1} = \mu$, and by the same method we can find $W' = w_{j_1} + \cdots + w_{j_s}$ which satisfies the same relations as Z' for the a set $F_1 \in Tan(F, x)$, we can suppose that $(E_1, F_1) \in Tan(E, F, \bar{x})$. Then by Lemma 3.3.3 we can find $\mathcal{H}^{Q-1}$-a.e. $x \in \partial^* E \cap \partial^* F$ a $(E_2, F_2) \in Tan(E_1, F_1, x)$ such that $v_{i_k}\chi_{E_2} = 0$ and $w_{j_s}\chi_{F_2} = 0$, i.e. $v_{i_k} \in Inv(E_2)$ and $w_{j_s} \in Inv(F_2)$. Since $v_{i_k} \notin Inv_0(E)$, $w_{j_s} \notin Inv_0(F)$ we have proved the Proposition. □

Remark 20. The condition on the codimension of $Inv(E)$ and $Inv(F)$ in V_1 will be automatically satisfied when dealing with tangent spaces, by Proposition 3.3.2.

Theorem 3.3.5. *Let E and F as above, then for a.e. $x \in \partial^* E \cap \partial^* F$*

$$(H_{0,\nu_E(x)}, H_{0,\nu_F(x)}) \in Tan^k(E, F, x) \quad with \quad k := 1 + 2(n - m).$$

Proof. First notice that a vertical halfspace H is invariant under dilation, this means that $\delta_\lambda(x^{-1}H) = H$ for all $\lambda > 0$.
By Proposition 3.3.2, sets in $Tan(E,F,x)$ are invariant in at least $m-1$ directions. Define the integers

$$i_k = \max\{\min\{dim(span(Inv_0(E_1))), dim(span(Inv_0(F_1)))\} \\ \textit{such that } (E_1,F_1) \in Tan^k(E,F,x)\}.$$

Then $i_1 \geq m-1$, and by Lemma 3.3.4 if follows that $i_{k+2} > i_k$ as long as there exist $(E_1,F_1) \in Tan^k(E,F,x)$ such that $dim(span(Inv_0(E_1))) \leq n-2$, and the same for F_1. On the other hand, if $dim(span(Inv_0(E_1))) \leq n-2$ and $dim(span(Inv_0(F_1))) \geq n-1$, for all $(E_1,F_1) \in Tan^k(E,F,x)$ we apply the original version of Lemma 3.3.4 (see [4]) to the first component, i.e. there exist an $E_1' \in Tan^{k+2}(E,x)$ such that $dim(span(Inv_0(E_1')) > i_k$. Indeed, by Proposition 4.4 in [4] and the monotonicity given by Proposition 5.4 in [4], $dim(span(Inv_0(F_1)) \geq n-1$ implies that F_1 is an halfspace, thus $(E_1',F_1) \in Tan^{k+2}(E,F,x)$ and $i_{k+2} > i_k$. Iterating the tangent operator k times with $k \leq 2(n-m)$, we can find $(Z_1,Z_2) \in Tan^k(E,F,x)$ with $dim(Inv_0(Z_j)) \geq n-1$, $j=1,2$. Note that with this procedure we add only vertical invariant directions, it follows that Z_1, Z_2 have at least codimension 1 so $dim(Inv_0(Z_j)) = n-1$ and they are monotone in the remaining direction by Proposition 3.3.2, thus they are vertical halfspaces and the theorem follows. □

Definition 3.11 (Tangents to a measure)**.** Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. We denote by $Tan(\mu,x)$ the family of all measures $\nu \in \mathcal{M}^m(\mathbb{G})$ that are weak* limit point as $r \downarrow 0$ of the family of measures $r^{-q}(I_{x,r})_\sharp\mu$. Where $I_{x,r}(y) := \delta_{1/r}(x^{-1}y)$.

Now, it remains to prove that $T^k(E,F,x) \subseteq Tan(E,F,x)$ for every $k \geq 2$. The following theorem is a variant of a theorem in Mattila's book [39], for a proof we refer to [4].

Theorem 3.3.6 ([4])**.** *Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. Then, for μ-a.e. x, the following property holds:*

$$Tan(\nu,y) \subseteq Tan(\mu,x) \qquad \forall \nu \in Tan(\mu,x),\ y \in supp|\nu|.$$

Theorem 3.3.7 ([4]). *Let $F, E \subseteq \mathbb{G}$ be sets with locally finite perimeter. Then for $\mathcal{H}^{Q-1}$-a.e. $x \in \mathbb{G}$ we have*

$$\bigcup_{k=2}^{\infty} Tan^k(E,F,x) \subseteq Tan(E,F,x).$$

Remark 21. The connection between the two previous theorem is the following: consider the vector-valued measure $(D\chi_E, D\chi_F) \in \mathcal{M}^{2m}(\mathbb{G})$ then

$$(E_1,F_1) \in Tan(F,E,x) \Longleftrightarrow (D\chi_{E_1}, D\chi_{F_1}) \in Tan(D\chi_E, D\chi_F, x) \setminus \{0\}.$$

Assume without loss of generality that $x = e$ and that $(D\chi_{E_1}, D\chi_{F_1}) \neq 0$ is the weak* limit of $(r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_E, r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_F)$, as $r_i \downarrow 0$. Set $E^i = \delta_{1/r_i}E$, $F^i = \delta_{1/r_i}F$, by the compactness properties of sets of finite perimeter we can assume that $(E^i, F^i) \to (E', F')$ locally in measure. Then $(r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_E, r_i^{1-Q}(I_{e,r_i})_\sharp D\chi_F) = (D\chi_{E^i}, D\chi_{F^i})$ weakly* converge to $(D\chi_{E'}, D\chi_{F'})$ it follows that $(D\chi_{E'}, D\chi_{F'}) = (D\chi_{E_1}, D\chi_{F_1})$. Since $\chi_{E_1} - \chi_{E'}$ has zero horizontal distributional derivative, it is equivalent to a constant, by (3.3.1) $Inv(\chi_{E_1} - \chi_{E'}) = \mathfrak{g}$. Using the method in the proof of Proposition 3.2.4 it follows that $\chi_{E_1} - \chi_{E'}$ is equivalent to a constant; this can appen only when $E_1 = E'$ or $E_1 = \mathbb{G} \setminus E'$ and the second possibility is ruled out because it implies $D\chi_{E_1} = 0$. Clearly the same holds also for F_1 and we have proved that $(E_1, F_1) \in Tan(E,F,e)$.
The converse implication follows easily by a scaling argument.

Proof. Let $x \in \mathcal{F}E \cap \mathcal{F}F$ be satisfying the property stated in Theorem 3.3.6 with $(\mu,\nu) = (D\chi_E, D\chi_F)$. Consider $(E_1,F_1) \in Tan(E,F,x)$ and $(E_2,F_2) \in Tan(E_1,F_1,y)$ for some $y \in \mathcal{F}E \cap \mathcal{F}F$. By Remark 21 we know that $(D\chi_{E_2}, D\chi_{F_2}) \in Tan(D\chi_E, D\chi_F, x) \setminus \{0\}$ then we have $(E_2,F_2) \in Tan(E,F,x)$, this ends the proof. □

Corollary 3.3.8. *Let $F \subset E \subseteq \mathbb{G}$ be sets of locally finite perimeter, then for $\mathcal{H}^{Q-1}$-a.e. $x \in \partial^*E \cap \partial^*F$ we have $\nu_E(x) = \nu_F(x)$.*

Proof. Consider the set $N \subset \partial^*E \cap \partial^*F$ where Theorem 3.3.5 holds and fix an $x \in N$, then by Theorem 3.3.7 there exist $(H_{0,\nu_E}, H_{0,\nu_F}) \in Tan(E,F,x)$. Notice that $F \subset E$ implies $H_{0,\nu_F} \subset H_{0,\nu_E}$, then working in exponential coordinates

$$\left\{\sum_{i=1}^m a^i X_i + Y,\ \sum a^i \nu_{E,i} \le 0\right\} \supset \left\{\sum_{i=1}^m b^i X_i + Y,\ \sum b^i \nu_{F,i} \le 0\right\} \tag{3.22}$$

where $Y \in \bigoplus_2^s V_j$, then if $\nu_E(x) \neq \nu_F(x)$ we can find $b = (b^1, \dots, b^m) \in \mathbb{R}^m$ such that

$$< b, \nu_F(x) >= 0 \qquad and \qquad < b, \nu_E(x) >= 1$$

and this is in contrast with (3.22). $\square$

In order to prove the locality property we need to extend the previous corollary to all possible sets $E, F \subset \mathbb{G}$ of locally finite perimeter :

Corollary 3.3.9. *Let $F, E \subseteq \mathbb{G}$ be sets of locally finite perimeter, then $\mathcal{H}^{Q-1}$-a.e. $x \in \partial^* E \cap \partial^* F$ we have $\nu_E(x) = \pm\nu_F(x)$.*

Proof. Consider the set $E \cap F$, clearly $E \cap F \subset F$ and the same for E, then by Corollary (3.3.8) we have

$$\nu_{E\cap F} = \nu_E \quad on\ \partial^*(E \cap F) \cap \partial^* E, \qquad \nu_{E\cap F} = \nu_F \quad on\ \partial^*(E \cap F) \cap \partial^* F.$$

Noticing that $\nu_E = -\nu_{E^c}$, then again by Corollary 3.3.8 we get

$$\nu_{E\setminus F} = \nu_E \quad on\ \partial^*(E \setminus F) \cap \partial^* E, \qquad \nu_{E\setminus F} = -\nu_F \quad on\ \partial^*(E \setminus F) \cap \partial^* F,$$

the same relations hold for $F \setminus E$.
It follows that $\nu_E = \pm\nu_F$ on $(\partial^*(E\cap F)\cup\partial^*(E\setminus F)\cup\partial^*(F\setminus E))\cap(\partial^*E\cap\partial^*F)$. It remains to show that this holds a.e. in $\partial^* E \cap \partial^* F$, write $E = (E \cap F) \cup (E \setminus F)$ then we have the inclusion

$$\partial^* E \subset \partial^*(E \cap F) \cup \partial^*(E \setminus F),$$

the same relation holds also for F. Therefore taking the intersection we get

$$\partial^* E \cap \partial^* F \subset \partial^*(E \cap F) \cap (\partial^*(E \setminus F) \cup \partial^*(F \setminus E)),$$

and the proof is complete. $\square$

Remark 22. Given a set $E \subset \mathbb{G}$, it is not difficult to show that $\partial^* E$ is a Borel set. Consider the map $F_\rho^E : x \mapsto \rho^{-Q} vol_{\mathbb{G}}(E \cap B_\rho(x))$ is lower semicontinuous, thus it is a Borel map. Notice that $\partial^* E$ is defined using F_ρ^E and taking liminf and limsup respect to ρ and these operations preserve the Borel property, thus $\partial^* E$ is a Borel set.

By the locality property of the outer normal and by a blow-up argument and a measure derivation we will prove that $\theta_E = \theta_F$ on $\partial^* E \cap \partial^* F$, see Theorem 1.3.1 for the definition of θ_E.

Theorem 3.3.10 (Locality property)**.** *Let $E, F \subseteq \mathbb{G}$ set of locally finite perimeter, then*

$$\theta_E = \theta_F \qquad \mathcal{H}^{Q-1} \text{ a.e. } \; x \in \partial^* E \cap \partial^* F.$$

Here θ_E, θ_F are the Borel functions that represent the perimeter measure by Theorem (1.3.1), i.e.

$$|D\chi_E|(A) = \int_{A\cap\partial^* E} \theta_E(x)\, d\mathcal{H}^{Q-1}, \qquad |D\chi_F|(A) = \int_{A\cap\partial^* F} \theta_F(x)\, d\mathcal{H}^{Q-1}.$$

Proof. If we prove that the density $K := |D\chi_E|/|D\chi_F|$ is constant and equal to 1 (in $\partial^* E \cap \partial^* F$) the theorem follows, however we need to clearly define this density. Indeed, $|D\chi_E|$ and $|D\chi_F|$ have different supports, but we can overcome this difficulty.
Let $x \in \partial^* E \cap \partial^* F$, noticing that $\partial^* E$ and $\partial^* F$ are Borel sets and using (3.11) one can prove that

$$\lim_{r\downarrow 0} \frac{|D\chi_E|((\partial^* E \setminus \partial^* F) \cap B_r(x))}{r^{Q-1}} = 0$$

in fact if we supposte that the above density is $> \epsilon$ on a Borel set $A \subset \partial^* E \cap \partial^* F$ with $\mathcal{S}^{Q-1}(A) > 0$, by (3.11), we have that

$$|D\chi_E|((\partial^* E \setminus \partial^* F) \cap A) \geq \epsilon \mathcal{S}^{Q-1}(A) > 0,$$

this is impossible because $(\partial^* E \setminus \partial^* F) \cap A = \emptyset$. Then write

$$\begin{aligned} |D\chi_E| &= |D\chi_E| \llcorner \partial^* E \cap \partial^* F + |D\chi_E| \llcorner \partial^* E \setminus \partial^* F = \mu_1 + \mu_2 \\ |D\chi_F| &= |D\chi_F| \llcorner \partial^* E \cap \partial^* F + |D\chi_F| \llcorner \partial^* E \setminus \partial^* F = \nu_1 + \nu_2. \end{aligned}$$

Notice that μ_1 is absolutely continuous with respect to ν_1 by the Ahlfors property (1.8) and both measure are asymptotically doubling, by Theorem 3.2.5. From the density estimates we have that $\mu_2(B_r(x)) = o(r^{Q-1})$, $\nu_2(B_r(x)) = o(r^{Q-1})$, then we can define $K = |D\chi_E|/|D\chi_F| := \mu_1/\nu_1$ for points $x \in \partial^* E \cap \partial^* F$.

Fix a point $x \in \mathcal{F}E \cap \mathcal{F}F$ such that $\nu_E(x) = \nu_F(x)$. Let $h \in C^1_c(\mathbb{G})$ a radial function with respect to a smooth distance equivalent to the CC-distance, $0 \leq h \leq 1$, $h = 0$ outside

$B_1(e)$ and suppose that $\int_{\mathbb{G}} h\, dvol_{\mathbb{G}} = 1$, consider the dilated and translated functions $h_{r,x}(y) = h(\delta_{1/r}(x^{-1}y))$. We have

$$K = \lim_{r\downarrow 0} \frac{\int_{\mathbb{G}} h_{r,x}\, d\,|D\chi_E|}{\int_{\mathbb{G}} h_{r,x}\, d\,|D\chi_F|} \tag{3.23}$$

this is a consequence of the following general fact, consider $f, g \in L^1(0,\infty)$ suppose that

$$\lim_{r\downarrow 0} \frac{f(r)}{g(r)} = L$$

then given a smooth function ρ of compact support such that $\int_0^\infty \rho = 1$ we have

$$\lim_{r\downarrow 0} \frac{\int_0^\infty f(y)\rho(\frac{y}{r})\, dy}{\int_0^\infty g(y)\rho(\frac{y}{r})\, dy} = L.$$

Define $f(s) = |D\chi_E|(B_s(x))$ and $g(s) = |D\chi_F|(B_s(x))$, by the symmetries of h we can write the right hand part of (3.23) as

$$\lim_{r\downarrow 0} \frac{\int_0^\infty h_{r,x}(s) f(s)\, ds}{\int_0^\infty h_{r,x}(rs) g(s)\, ds}$$

and the thesis follows from the previous argument. Now, write $|D\chi_E| = \langle \nu_E(\cdot), d\, D\chi_E\rangle$ and the same for F, thus one get

$$K = \lim_{r\downarrow 0} \frac{\int_{\mathbb{G}} \langle \nu_E(y) h_{r,x}(y), d\, D\chi_E\rangle}{\int_{\mathbb{G}} \langle \nu_F(y) h_{r,x}(y), d\, D\chi_F\rangle}$$

We would like to integrate by parts, so first we have to exchange the $\nu_E(y)$ with the constant $\nu_E(x)$ and the same for F. This does not change the limit, indeed we have

$$\begin{aligned}\left|\int_{\mathbb{G}} \langle (\nu_E(y) - \nu_E(x)) h_{r,x}(y), d\, D\chi_E\rangle\right| &\le \int_{B_r(x)} \langle \nu_E(y) - \nu_E(x), \nu_E(y)\rangle\, d\,|D\chi_E| \\ &= |D\chi_E|(B_r(x)) - \left\langle \nu_E(x), \int_{B_r(x)} \nu_E(y) d\,|D\chi_E| \right\rangle = o(r^{Q-1}).\end{aligned}$$

Then we have

$$K = \lim_{r\downarrow 0} \frac{\int_{\mathbb{G}} \langle \nu_E(x) h_{r,x}(y), d\, D\chi_E\rangle}{\int_{\mathbb{G}} \langle \nu_F(x) h_{r,x}(y), d\, D\chi_F\rangle}$$

and integrating by parts

$$= \lim_{r\downarrow 0} \frac{\int_E \langle \nu_E(x), \nabla_H h_{r,x}(y)\rangle\, d\, vol_{\mathbb{G}}}{\int_F \langle \nu_F(x), \nabla_H h_{r,x}(y)\rangle\, d\, vol_{\mathbb{G}}}$$

where ∇_H is the horizontal gradient, i.e. given a $u \in C^1(\mathbb{G})$, $\nabla_H u = (X_1 u, \dots, X_m u)$. Now, by a change of variable we get

$$K = \lim_{r \downarrow 0} \frac{\int_{\delta_{1/r}(x^{-1}E)} \langle \nu_E(x), \nabla_H h(y) \rangle \, d\, vol_{\mathbb{G}}}{\int_{\delta_{1/r}(x^{-1}F)} \langle \nu_F(x), \nabla_H h(y) \rangle \, d\, vol_{\mathbb{G}}}.$$

Since K is a limit, we can chose a sequence $r_i \to 0$ and compute the limit of the previous equation as $i \to \infty$.
Choosing r_i such that $(\delta_{1/r}(x^{-1}E), \delta_{1/r}(x^{-1}F)) \to (H_{0,\nu_E(x)}, H_{0,\nu_F(x)})$, if $\nu_E(x) = \nu_F(x)$ it follows that $K = 1$. If $\nu_E(x) = -\nu_F(x)$, using that h is a function of compact support we have

$$\int_{\mathbb{G}} \langle \nu_E(x), \nabla_H h(y) \rangle \, d\, vol_{\mathbb{G}} = 0$$

then

$$\int_{H_{\nu_E(x)}} \langle \nu_E(x), \nabla_H h(y) \rangle \, d\, vol_{\mathbb{G}} + \int_{H_{-\nu_E(x)}} \langle \nu_E(x), \nabla_H h(y) \rangle \, d\, vol_{\mathbb{G}} = 0$$

and $K = 1$.
Therefore, we know that $|D\chi_E| = |D\chi_F|$, $\mathcal{H}^{Q-1}$-a.e. in $\partial^* E \cap \partial^* F$ and the theorem follows by the representation formula of the perimeter measure (1.2.2). $\square$

Notice that given $u \in BV(\mathbb{G})$ and $\psi \in \Lambda$, where Λ is defined in (1.38), by Theorem 1.3.5 the measure $|D\psi \circ u|$ satisfies

$$|D(\psi \circ u)| = \psi'(\tilde{u})|D^d u| + \Psi(u) S^h \llcorner S_u$$

where Ψ is defined as follows (see 1.41)

$$\Psi(u)(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(t) \theta_{\{u>t\}}(x) dt$$

Now, we know that $\mathbb{G}$ is a $\mathcal{U}$-space in the sense of Definition (1.11) then we have by (1.3.10) the following Proposition

Proposition 3.3.11. *Let $\mathbb{G}$ a Carnot group and let $u \in BV(\mathbb{G})$ with $H^{Q-1}(S_u) < \infty$. Then, there is a function $\Theta_u : S_u \to [\alpha, C_D]$ such that*

$$\Psi(u) = [\psi(u^\vee) - \psi(u^\wedge)] \, \Theta_u, \tag{3.24}$$

for every $\psi \in \Lambda$, where α is the constant in Theorem 1.3.1.

As a consequence of the previous Proposition, if $u \in SBV(\mathbb{G})$, see (1.10) for the definition, and let $\psi \in \Lambda$ then

$$|D(\psi \circ u)| = \psi'(\tilde{u})|Gu|\, vol_{\mathbb{G}} + [\psi(u^{\vee}(x)) - \psi(u^{\wedge}(x))]\, \Theta_u \mathcal{H}^{Q-1} \llcorner S_u.$$

We can now formulate, as an application of the locality property, the generalized Munford-Shah functional in $\mathbb{G}$ and state the existence theorem of SBV minimizers.

Theorem 3.3.12 ([2])**.** *Let $\mathbb{G}$ a Carnot group, $\in L^{\infty}(\mathbb{G})$, $p > 1$, $q > 0$. Then, there exists a minimizer of the functional*

$$F(u) = \int_{\mathbb{G}} |Gu|^p \, d\, vol_{\mathbb{G}} + \alpha \int_{\mathbb{G}} |g - u|^q \, d\, vol_{\mathbb{G}} + \beta \int_{S_u} \Theta_u d\, \mathcal{H}^{Q-1} \qquad u \in SBV(\mathbb{G}).$$

3.4 Rectifiability

The notion of rectifiability is central to the study of geometric measure theory, recently there has been progress in the study of rectifiable sets in non Euclidean spaces, see [3], [4], [10], [15], [23], [33].
In order to investigate the properties of rectifiable sets in Carnot groups, here we consider a subset to be rectifable if it can be realized as the image of a Lipschitz map of a piece of Euclidean space, one encounters a difficulty: there may not be any rectifiable subset (see [3], [37]). Therefore we need a more general notion of rectifiable sets, in this section we will concentrate our attention to this problem in the Heisenberg group $\mathbb{H}^n$: following [23], we define H-regular surfaces and we give an intrinsic definition of rectifiability with images of $C^1_{\mathbb{H}}$ functions.

First we give some examples of pure unrectifiability sets in the case of Heisenberg group and a generalization of it found by V. Magnani [37], this will justify the intrinsic definition of rectifiability in the Heisenberg group, given in the last part of the section ([23]).

3.4.1 Calculus in Carnot groups

Here we give the definition of metric and Pansu differentiability (see [44]), after that we state a Rademacher type result and a general coarea formula.

A map $L : \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-linear if it is a homomorphism from $\mathbb{G} = (\mathbb{R}^n, \cdot)$ to $(\mathbb{R}, +)$ and L is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{G}$. We indicate as $\mathcal{L}_{\mathbb{G}}$ the set of $\mathbb{G}$-linear functionals, this space is endowed with the norm

$$\|L\|_{\mathcal{L}_{\mathbb{G}}} := \sup\{|L(p)| : d(p, 0) \le 1\}.$$

Recall that $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, where $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$. Choose graded coordinates such that we can identify $\mathbb{G}$ with $\mathbb{R}^n$ and set $x^j = (x_{h_{j-1}+1}, \dots, x_{h_j}) \in \mathbb{R}^{m_j}$ for $1 \le h \le s$, where $h_j = \sum_1^j m_i$.

Proposition 3.4.1 ([24])**.** *A map $L : \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-linear if and only if there is $a = (a_1, \dots, a_{m_1}) \in \mathbb{R}^{m_1}$ such that, if $x = (x_1, \dots, x_n) \in \mathbb{G}$, then $L(x) = \sum_{i=1}^{m_1} a_i x_i$.*

Proof. Clearly any function L of the form $L(v) = \sum_{i=1}^{m_1} a_i v_i$ is $\mathbb{G}$-linear. Conversely, let L be $\mathbb{G}$-linear and write

$$x = (x_1, \dots, x_n) = [x^1, \dots, x^s] \in \mathbb{R}^n.$$

Observe that if $1 < j \le s$ and $x = [0, \dots, x^j, \dots, 0]$, then

$$2Lx = L(x \cdot x) = L(\delta_{2^{1/\alpha_j}} x) = 2^{1/\alpha_j} Lx,$$

where $\alpha_j = k$ if $h_{k-1} + 1 < j \le h_k$, thus $Lx = 0$. From the grading structure and the product law (3.1.1)

$$x = \left[0, x^2, \dots, x^s\right] = \left[0, x^2, 0, \dots, 0\right] \cdot \left[0, 0, y^3, y^4 \dots, y^s\right]$$

for appropriate $y^3, \dots, y^s$. Hence we have

$$L\left[0, x^2, \dots, x^s\right] = L\left[0, x^2, 0, \dots, 0\right] + L\left[0, 0, y^3, y^4 \dots, y^s\right] = L\left[0, 0, x^3, y^4 \dots, y^s\right],$$

iterating this procedure we get $Lx = 0$. Therefore we have proved that if $x = [0, x^2, \dots, x^s]$ it follows $Lx = 0$. Concluding, $x = [x^1, \dots, x^s] = [x^1, 0, \dots, 0] \cdot [0, y^2, \dots, y^s]$. Then the thesis follows from the representation of linear functions in Euclidean spaces. $\square$

Definition 3.12. Let Ω be an open set in $\mathbb{G}$, then $f : \Omega \to \mathbb{R}$ is Pansu-differentiable (see [44]) at x_0 if there is a $\mathbb{G}$-linear map L such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x_0^{-1} \cdot x)}{d(x, x_0)} = 0.$$

Remark 23. There is an equivalent definition of Pansu differentiability: there exist an homomorphism L from $\mathbb{G}$ to $(\mathbb{R}, +)$ such that

$$\lim_{\lambda \to 0^+} \frac{f(\tau_{x_0}(\delta_\lambda v)) - f(x_0)}{\lambda} = L(v)$$

uniformly with respect to v belonging to compact sets in $\mathbb{G}$. L is unique and we write $L = d_{\mathbb{G}} f(x_0)$.

Definition 3.13. Let E be a metric space, we say that a function $f : \mathbb{R}^k \to E$ is metrically differentiable at $x \in \mathbb{R}^k$ if there exist a seminorm $\| \cdot \|_x$ in $\mathbb{R}^k$ such that

$$d(f(y), f(x)) - \|y - x\|_x = o(|y - x|).$$

This seminorm will be said to be the metric differential and will be denoted by $mdf(x)$.

Theorem 3.4.2. *Any Lipschitz function $f : \mathbb{R}^k \to E$ is metrically differentiable at $\mathcal{L}^k$-a.e. $x \in \mathbb{R}^k$.*

Definition 3.14. Let V, W be Banach spaces, $L : W \to V$ linear. If $k = \dim W$ is finite, the k-jacobian of L is defined by

$$\mathbf{J}_k(L) := \frac{\omega_k}{\mathcal{H}^k\left(\{x \ : \ \|L(x)\| \leq 1\}\right)}.$$

If s is a seminorm in $\mathbb{R}^k$ we define also

$$\mathbf{J}_k(s) := \frac{\omega_k}{\mathcal{H}^k\left(\{x \ : \ s(x) \leq 1\}\right)}.$$

Theorem 3.4.3. *Let $f : \mathbb{R}^k \to E$ be a Lipschitz function. Then*

$$\int_{\mathbb{R}^k} \theta(x) \boldsymbol{J}_k(mdf_x) dx = \int_E \sum_{x \in f^{-1}(y)} \theta(y) d\mathcal{H}^k(y)$$

for any Borel funtion $\theta : \mathbb{R}^k \to [0, \infty]$ and

$$\int_A \theta(x) \boldsymbol{J}_k(mdf_x) dx = \int_E \theta(y) \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^k(y)$$

for $A \in \mathcal{B}(\mathbb{R}^k)$ and any Borel function $\theta : E \to [0, \infty]$.

3.4.2 Pure k-unrectifiability

Definition 3.15. We say that a Borel set $S \subset E$ is countably $\mathcal{H}^k$-rectifiable if there exist a sequence of Lipshitz functions $f_j : A_j \subset \mathbb{R}^k \to E$ such that $\mathcal{H}^k(S \setminus \bigcup_j f_j(A_j)) = 0$. We say that $\mu \in \mathcal{M}(E)$ is k-rectifiable if $\mu = \theta \mathcal{H}^k \llcorner S$ for some countably $\mathcal{H}^k$-rectifiable set S and some Borel function $\theta : S \to (0, \infty)$.

We give now and example of purely k-unrectifiable metric space, i.e. a metric space E such that $\mathcal{H}^k(S) = 0$ for any countably $\mathcal{H}^k$-rectifiable set $S \subset E$.

Theorem 3.4.4. *The Heisenberg group is purely k-unrectifiable for $k = 2, 3, 4$.*

Proof. Let $f : A \subset \mathbb{R}^k \to \mathbb{H}$ be a Lipschitz map and let us prove that $\mathcal{H}^k(f(A)) = 0$. Since $\mathbb{H}$ is complete we can assume with no loss of generality that A is closed. By the area formula we need to check that $\mathbf{J}_k(mdf_x) = 0$ at any metric differentiability point where the Pansu differential df_x is defined. Since $df_x(\mathbb{R}^k)$ is a commutative subgorup of H, it must be contained in $\mathbb{R}z_0 \times \mathbb{R}$ for some $z_0 \in \mathbb{C}$. Writing $df_x(v) = (z(v), t(v))$, the inequality

$$|t(v) - t(v')| \leq [Lip(df_x)]^2 \, |v - v'|^2 \qquad \forall v, \ \forall v' \ \in \ \mathbb{R}^k,$$

implies that t is constant, hence the image of df_x is contained in $\mathbb{R}z_0 \times \{0\}$ and the kernel of df_x has dimension al least $k - 1 \geq 1$. Since

$$\begin{aligned} mdf_x(v) &= \lim_{t \downarrow 0} \frac{d(f(x + tv), f(x))}{t} \\ &= \lim_{t \to 0} \| \left([f(x)]^{-1} f(x + tv) \right) \| = \|df_x(v)\| \end{aligned}$$

for any $v \in \mathbb{R}^k$ we conclude that

$$\mathcal{H}^k \left(\left\{ v \in \mathbb{R}^k \ : \ mdf_x(v) \leq 1 \right\} \right) = \infty$$

and hence $\mathbf{J}_k(mdf_x(v)) = 0$. $\square$

Consider two Carnot groups $\mathbb{M}$ and $\mathbb{G}$, S.D. Pauls in [45] proposed the following notion of rectifiablity: a subset E in $\mathbb{M}$ is $\mathbb{F}$-rectifiable if it is the image of a Lipschitz map defined on a subset of $\mathbb{F}$, where $\mathbb{F}$ is a subgroup of a stratified group $\mathbb{G}$. Note that if $\mathbb{G} = \mathbb{R}^n$ this definition of rectifiability is equivalent to (3.15). A result similar to the previous one, i.e. the pure unrectifiability, holds also with this definition of rectifiability (see [37]).

We state now the main theorems of the work [37], for the proofs we refer to the original paper. We begin with an algebraic characterization of all purely k-unrectifiable stratified groups.

Theorem 3.4.5 ([37]). *Let $\mathbb{M}$ be as stratified group with Lie algebra $\mathcal{M} = W_1 \oplus \cdots \oplus W_l$. Then $\mathbb{M}$ is purely k-unrectifiable if and only if there do not exist k-dimensional Lie subalgebras contained in the first layer W_1.*

The following theorem is a consequence of the area formula for Lipschitz mappings between stratified groups (see [35]) that follows from a Rademacher type theorem to stratified groups obtained by Pansu in [44]. Clearly, Theorem 3.4.6 yields Theorem 3.4.4 if $\mathbb{G} = \mathbb{R}^n$ and $\mathbb{M} = \mathbb{H}$.

Theorem 3.4.6 ([37]). *Let $\mathbb{M}$ and $\mathbb{G}$ be stratified groups with Lie algebras $\mathcal{M}$ and $\mathcal{G}$, respectively. Then $\mathbb{M}$ is purely $\mathbb{G}$-unrectifiable if and only if $\mathcal{M}$ does not contain any Lie subalgebra which is G-isomorphic to $\mathcal{G}$. Recall that two Lie algebras of stratified groups are G-isomorphic if there exist an algebra isomorphism that respects the grading.*

The last theorem in [37] is an improved version of the rigidity result proved in Theorem 3 of [44]: two biLipschitz equivalent stratified groups are G-isomorphic.

Theorem 3.4.7 ([37]). *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups and let $A \subset \mathbb{G}$ be a subset of positive measure. If there exists a biLipschitz mapping $f : A \to \mathbb{M}$, then $\mathbb{G}$ and $\mathbb{M}$ are G-isomorphic.*

3.4.3 $\mathbb{H}$-regular surfaces in the Heisenberg group

As we have seen in the previous section, images of Eclidean sets via Lipschitz maps are not good surfaces to consider in order to define rectifiability. Now, by means of $C^1_{\mathbb{H}}$ maps we define intrinsic regular hypersurfaces and with them an intrinsic notion of rectifiability. In this section we concentrate our attention only to the Heisenberg groups $\mathbb{H}^n$ where these definitions were first introduced by B. Franchi, R. Serapioni, F. Serra Cassano in [23], [24], [25].

Consider the Heisenberg group $\mathbb{H}^n$, the associated vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ in $\mathbb{R}^{2n+1}$ and the horizontal sub-bundle $H\mathbb{H}^n$ of $T\mathbb{H}^n$ generated by these vector fields.

Given $p = (x, y, t) \in \mathbb{H}^n$ set $\|p\|_\infty := \max\left\{|(x, y)|, |t|^{1/2}\right\}$, then for $p, q \in \mathbb{H}^n$ we define the homogeneus distance

$$d_\infty(p, q) = \|p^{-1} \cdot q\|_\infty.$$

Defining, as usual, the CC distance d_c as the distance associated to the vector fields X_i, Y_i we have the equivalence of d_c and d_∞.

Proposition 3.4.8. *The Carnot-Carathéordory distance d_c is equivalent to the distance d_∞.*

Remark 24. The previous proposition is a particular application of a more general result. A distance d on $\mathbb{G}$ is called homogeneus if it is invariant under left translations and for every $r > 0$

$$d(\delta_r x, \delta_r y) = r d(x, y).$$

One can prove that all homogeneus distances are equivalent. Note that the CC-distance is an homogeneus distance.

We say that f is differentiable along X_j (Y_j) at p if the map $\lambda \mapsto f(p \cdot \delta_\lambda e_j)$ ($\lambda \mapsto f(p \cdot \delta_\lambda e_{j+n})$) is differentiable at $\lambda = 0$, here we have identified $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and e_j is the j-th vector of the canonical basis of $\mathbb{R}^{2n+1}$.
If f is differentiable along X_j and Y_j at p for $j = 1, \ldots, n$, we define

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^{n} (X_j f) X_j + (Y_j f) Y_j$$

or

$$\nabla_{\mathbb{H}} f = (X_1 f, \ldots, X_n f, Y_1 f, \ldots, Y_n f)$$

in canonical coordinates.

Remark 25. $C^1(\Omega) \subset C^1_{\mathbb{H}}(\Omega)$, one can also prove that the inclusion is strict, see [23]. Moreover it is possible to prove that $C^1_{\mathbb{H}}$ functions are Lipschitz with respect to the distance d_∞.

Definition 3.16. If $\Omega \subset \mathbb{H}^n$ we denote by $C^1_{\mathbb{H}}(\Omega)$ the set of continuous real valued functions in Ω such that $\nabla_{\mathbb{H}} f$ is continuous in Ω.

Proposition 3.4.9 ([23]). *A continuous function belongs to $C^1_{\mathbb{H}}(\Omega)$ if and only if the distributional derivatives $X_j f$, $Y_j f$ are continuous in Ω for $j = 1, \dots, n$.*

Definition 3.17. $S \subset \mathbb{H}^n$ is a $\mathbb{H}$-regular hypersurface if for every $p \in S$ there exist an open ball $B(p, r)$ and a function $f \in C^1_{\mathbb{H}}(B(p, r))$ such that

$$S \cap B(p, r) = \{q \in B(p, r) : f(q) = 0\} \qquad \nabla_{\mathbb{H}} f(p) \neq 0.$$

We denote by $\nu_S(p)$ the horizontal normal to S at p, i.e. the vector

$$\nu_S(p) = -\frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|_p}.$$

One can prove that $\nu_S(p)$ is well defined, i.e. it does not depend on the choice of f, moreover ν_S is continuous.

Remark 26 ([23]). The classes of Euclidean regular hypersurfaces and $\mathbb{H}$-regular hypersurfaces are different: in $\mathbb{H}^1 \simeq \mathbb{R}^3$ consider the euclidean plane $T := \{(x, y, t) | \quad t = 0\}$, it is a smooth Euclidean submanifold but it is not $\mathbb{H}$-regular at the origin. Indeed one can prove that at the origin ν_T is not continuous. On the other side, a computation shows that $\Gamma := \left\{(x, y, t) : x - \sqrt{x^4 + y^4 + t^2} = 0\right\}$ is a $\mathbb{H}$-regular hypersurface but it is not an Euclidean C^1 submanifold at the origin.

Remark 27. We have seen that H-regular suface can behave quite badly from the Euclidean viewpoint, neverthless they are regular respect to the intrinsic geometry of $\mathbb{H}^n$, so they play the role of C^1 hypersurfaces in Euclidean spaces. Conversely one can prove that a C^1 regular hypersurface S is H-regular provided it has no characteristic points, i.e. at each point $p \in S$ the tangent plane at S does not coincides with the horizontal fiber $H_p\mathbb{H}$.

Definition 3.18. We say that $\Gamma \subset \mathbb{H}^n$ is $\mathbb{H}$-rectifiable it there exists a sequence of $\mathbb{H}$-regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ such that

$$\mathcal{H}^{Q-1}\left(\Gamma \setminus \bigcup_{j \in \mathbb{N}} S_j\right) = 0.$$

Now, we state the main result of [23], where it is proved that the measure theoretic boundary of a locally finite perimeter subset of $\mathbb{H}^n$ is $\mathbb{H}$-rectifiable.

Theorem 3.4.10 ([23]). *If $E \subseteq \mathbb{H}^n$ is a set of locally finite perimeter then*

$$\partial^* E \; is \; \mathbb{H} - rectifiable,$$

i.e. $\partial^* E = N \cup \bigcup_{i=1}^{\infty} K_i$, *where* $\mathcal{H}^{2n+1}(N) = 0$ *and* K_i *is a compact subset of a* $\mathbb{H}$-*regular surface* S_i, $\nu_E(p)$ *is* $\mathbb{H}$-*normal to* S_i *at* p, $\forall p \in K_i$, *that is* $\nu_E(p) \in H_p\mathbb{H}^n$ *and* $< \nu_E(p), v >_p = 0$ *for all* $v \in T_{\mathbb{H}} S_i(p)$. *Moreover we have the following representation of the perimeter measure with respect to the spherical Husdorff measure* $\mathcal{S}^{2n+1}$

$$|D\chi_E| = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \mathcal{S}^{2n+1} \llcorner \partial^* E.$$

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